Nārāyaṇa Paṇḍita's

Bījagaṇitāvataṃsa

["The Crown of Algebra"]

With English Translation
Explanation And Notes

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[“The Crown of Algebra”]

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Foreword

The present work, *Bījagaṇitāvataṃsa,* of Nārāyaṇa Paṇḍita contains the Sanskrit text with Introduction, English translation, exposition, notes, rationales of the rules, and complete solution of illustrative examples according to the methods of Nārāyaṇa using modern symbols.

Nārāyaṇa Daivajña or Nārāyaṇa Paṇḍita as he was popularly called, was the son of Nṛsiṁha Daivajña. He is solely a mathematician and the author of two works namely (i) Gaṇitakaumudī (G.K.) a mathematical treatise in 14 chapters, and (ii) Bījagaṇitāvataṃsa a work on algebra. Besides these two works, Nārāyaṇa has also written a work on Sanskrit prosody viz. Maṇimañjarī on Sanskrit prosody. Nārāyaṇa Paṇḍita lived in the fourteenth century of the Christian era.

Bījagaṇitāvataṃsa is a mathematical work, written in Sanskrit, by Nārāyaṇa Paṇḍita (1356 C.E.). The name *Bījagaṇitāvataṃsa* is a compound formed by the composition of the words *bījagaṇita,* meaning "algebra" (lit. "the science of calculation with elements" or the science of analytical calculation.") and *avataṃsa*, meaning "a garland or any ring-shaped ornament", or "the crown." *Bījagaṇitāvataṃsa* thus means "a garland of the elements of algebra" or "**the crown of algebra.**"

The present name for the science of *Bījagaṇita* in English is algebra. Algebra is derived from *al-jabr*

occurring in the title of an Arabic book on Algebra by al-Khwārizmī (died about 860 A.D.). As an operation, *jabr* denotes the transformation of a subtracted term to the other side of an equation and was used along with the three other operations, called *mukhābalah, radd*, and *ikmāl* (or *takmīl*), for solving algebraic equations.

The early (elementary) algebra was mostly the study of equations and methods for solving them while the modern (abstract) algebra is the study of mathematical structures, such as groups, rings, and fields.

Usually Hindu works on algebra, is divided into two parts, Part I dealing with algebraic-processes essential in solving algebraic equations (*bījōpayōgī-gaṇita*) and Part II dealing with algebraic equations (*bīja*).

The Sanskrit text adopted here, is as found in the printed form, edited by Kripa Shankar Shukla which is based on an incomplete manuscript containing the whole of Part I and a few opening lines of part II, (and is incomplete).

Part I sets forth the following three topics :

(1) Algebraic operations (of addition, subtraction, multiplication, division, squaring and extracting the square-root) for each of the following :

(i) Positive and negative numbers (*dhanarṇa*),

(ii) The zero (*śūnya*),

(iii) Single unknown (*avyakta*),

(iv) More unknowns characterized by colours (*varṇa*),

(v). surds (*karaṇī*).

(2) The Pulveriser (*kuṭṭaka*), i.e. analysis pertaining to the indeterminate equation $(ax \pm c)/b = y$.

(3) The Square-nature (*varga-prakṛti*), i. e., analysis pertaining to the indeterminate equation :

$$Nx^2 \pm c = y^2.$$

The use of symbols - letters of the alphabet to denote unknowns - and equations are the foundations of the science of algebra. The Hindus were the first to make systematic use of the letters of the alphabet to denote unknowns. They were also the first to classify and make a detailed study of equations, Thus they may be said to have given birth to the modern science of algebra.

The Indian method of writing equations was better than the Chinese, and in one respect was the best that has ever been suggested. The method is a fine example of syncopated (or semi-symbolic) algebra and the style was uniformly used for more than a thousand years.

Indian algebra is especially noteworthy for its development of indeterminate analysis.

The famous Indian *Cakravāla* or Cyclic method for solving $Nx^2 + 1 = y^2$ was known to Jayadeva (c,1000 A.D.) or possibly to earlier Indians and is considered to be "certainly the finest thing which was achieved in the theory of numbers before Lagrange" (c. 1780).

A study of this book, *Bījagaṇitāvataṃsa*, will reveal to the reader the remarkable progress in algebra made by the Hindus at an early date. It will also show that we are

indebted to the Hindus for the technique and fundamental results of algebra just as we owe to them the place-value system in arithmetic, and invention of the basic function 'sine' in Trigonometry.

Engineer Venugopal D. Heroor is a very keen, enthusiastic and devoted scholar of Indian mathematics. He has already benefitted students, teachers and other scholars by his "*The History of Mathematics & Mathematicians of India*", "*Bhāratīya Trikoṇamiti Śāstra*", "*Brahmagupta Gaṇitaṁ*", "*Pāṭīgaṇita Sāra*", *Bhāskrīya Bījagaṇitam* and several other books.

Readers (teachers, students, practitioners, historians etc.) will definitely find Er. Heroor's present book, *Bījagaṇitāvataṃsa* also both charming and exciting as well as enlightening.

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Preface

The present work contains the Sanskrit text of *Bījagaṇitāvataṃsa of* Nārāyaṇa Paṇḍita with Introduction, English translation, exposition, notes, rationales of the rules, and complete solution of illustrative examples according to the methods of Nārāyaṇa using modern symbols.

Bījagaṇitāvataṃsa ["The Crown of Algebra"] is a mathematical work, written in Sanskrit, by Nārāyaṇa Paṇḍita(1356 A.D.). A crtical edition of the work, availabale now, in printed form, edited by Kripa Shankar Shukla and published by Akhila Bharatiya Sanskrit Parishad, Lucknow, in 1970, is based on an incomplete manuscript containing the whole of Part I and a few opening lines of part II.

Like other Hindu works on algebra, it is divided into two parts, Part I dealing with algebraic-processes essential in solving algebraic equations *(bījjopayogi-gaṇita)* and Part II dealing with algebraic equations (*bīja*).

Part I sets forth the following three topics :

(1) Algebraic operations (of addition, subtraction, multiplication, division, squaring and extracting the square-root) for each of the following :

(i) Positive and negative numbers (*dhanarṇa*),

(ii) The zero (*śūnya*),

(iii) Single unknown (*avyakta*),

(iv) More unknowns characterized by colours (*varṇa*),

(v). surds (*karaṇī*).

The method for finding the square root of a surd of the type $(a+\sqrt{b}+\sqrt{c}+\sqrt{d}\ )$ as given in Rule 46-49,[1] and the method of finding the approximate values of the square-root of a non square number (quadratic surd) as given in rule 86, were given by Nārāyaṇa Paṇḍita for the first time; and are note worthy.

(2) The Pulveriser (*kuṭṭaka*), i.e. analysis pertaining to the indeterminate equation : $(ax \pm c)/b = y$.

Though the chapter on pulveriser in both the books *Gaṇitakaumudī* and *Bījagaṇitāvatāṃsa,* by the same author, is nearly the same, two rules [for the solution of the pulveriser when the solution is fractional viz. rule 68 and 69 **],** an example [Ex. 36] and a comment available in this book is not available in the former one. A set of four Sūtras [R.32-36(i)], followed by six examples (Ex.30-35) dealing with the residual and conjunct pulverisers *sāgra-kuṭṭaka and saṁśliṣṭa- kuṭṭaka* occuring (between verses 64 and 65) in *Gaṇitakaumudī* are not found here in *Bījagaṇitāvatāṃsa*. So they have been included here in Appendix-1.

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[1] . "SURDS IN HINDU MATHEMATICS" by Datta and Singh revised by K.S. Shukla IJHS. 28 (3), 1993 Pp. 254-264. see Pp. 263.

(3) **The Square-nature (*varga-prakṛti*)**, i. e., analysis pertaining to the indeterminate equation : $Nx^2 \pm c = y^2$.

It is mentioned by the editor, that the manuscript breaks off after giving the list of contents of Part II and an example of a linear equation in one unknown.

The Sanskrit text as found in the printed form, edited by Kripa Shankar Shukla (mentioned above) is adopted here. In the translation, the portion of English translation of verses available in the *History Of Hindu Mathematics part -II* (*Algebra*) by Bibhutibhusan Datta and Avadhes Narayan Singh is reproduced as it is. Occasionally, English translation of *Gaṇitakaumudī* by Paramananad Singh have also been reffered. Regarding the explanation part and rationale of rules some intermediary steps are added wherever found necessary. Complete solution of illustrative examples according to the methods of Nārāyaṇa Paṇḍita using modern symbols is given and the answers obtained are verified to facilitate the readers in general, students and Sanskrit scholars with elementary knowledge of mathematics in particular.

Parallel rules and similar examples found in other available works on Hindu mathematics have been indicated in the translation and foot notes. Heading and sub-headings have been provided to facilitate consultation.

I have been indebted to and relied on the expository source works by great savants like Bibhutibhushan Datta, A.N. Singh, K.S. Shukla, and Paramananad Singh. Works

of Datta and Singh have been the main source of inspiration for me in this work. I express my sincere gratitude to all those stawarts in the field.

I am grateful to Prof. Shrinivasa Varkhedi, VC, for gracing this book with his valuable foreword and also thanks to Prof. Madhusudan Penna, Director, Publication, Kavikulaguru Kalidas Sanskrit University for including this book in University Publications.

I express my thanks and appreciation to Dr. Renuka Bokare, PRO for publication support and Shri. Umesh Patil, Sr. Clerk, KKSU for the cover page design.

I express my profound gratitude to authorities of Kavikulaguru Kalidas Sanskrit University for taking up the task of publishing the work.

The author hopes that this book will be quite useful and interesting to the students and researchers in the field of Indian Mathematics.

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ABBREVIATIONS

Ā	*Āryabhaṭīya*
ĀBh.	*Āryabhaṭīya Bhāskarīya Bhāṣya*
BM	*Bakhshālī Manuscript*
BBi (ASS)	*Bhāskara II's Bījagaṇita(Ānandāśrama Sanskrit Series)*
BBi (HTC)	*Bhāskara II's Bījagaṇita (English Translation by H T Colebrooke)*
BrSpSi	*Brāhmasphuṭa siddhānta*
GK	*Gaṇitakaumudī*
GSS	*Gaṇita-Sāra-Saṅgraha*
GT	*Gaṇita-tilaka*
HHM-I	*History of Hindu Mathematics (Arithmetic)*
HHM-II	*History of Hindu Mathematics (Algebra)*
IJHS	*Indian Journal of History of Science*
L(ASS)	*Līlāvatī (Ānandāśrama Sanskrit Series)*
LBh.	*Laghubhāskarīya*
MBh.	*Mahābhāskarīya*
MSi	*Mahā-siddhānta*
NBi	*Bījagaṇitāvataṃsa*
PG	*Pāṭīgaṇita*
Si.Śe	*Siddhānta- Śekhara*
Si.Śi.	*Siddhānta- Śiromaṇi*
Si.TVi.	*Siddhāntatattvaviveka*

Introduction

Nārāyaṇa Paṇḍita (A.D. 1356) :

Nārāyaṇa Daivajña or Nārāyaṇa Paṇḍita as he was popularly called, was the son of Nṛsiṁha Daivajña. He is solely a mathematician and the author of two works namely (i) *Gaṇitakaumudī*[1] *(G.K.)* a mathematical treatise in 14 chapters, and (ii) *Bījagaṇitāvataṃsa*[2] a work on algebra. Besides these two works, Nārāyaṇa has also written a work on Sanskrit prosody viz. *Maṇimañjarī*[3] on Sanskrit prosody.

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1. Padmakar Dvivedi: *Gaṇitakaumudī* (GK) : in two parts, The Princess of Wales, Saraswati, Bhavan Texts, No.57, Benaras, Part-I, 1936 and Part –II, 1942.

   Paramanand Singh: *The Gaṇitakaumudī of Narayana Pandita* English translation with notes. Ch. I to III. G.B.Vol.20. Nos:1-4 (1998) Pp.25-82.
   Ch.IV. *Gaṇita Bhāratī (GB)* . Vol.21. Nos.1-4 (1999) P.10-73.
   Ch.V-XII *Gaṇita Bhāratī (GB)* Vol.22. No:1-4 (2000). Pp. 19-85
   Ch.XIII *Gaṇita Bhāratī (GB)*.Vol.23 Nos:1-4. Pp.18-82.
   Ch.XIV. *Gaṇita Bhāratī (GB)*.Vol.24. Nos:1-4 (2002) pp.35-98

   Takanori Kusuba, 'Combinatorics and Magic Squares in India' A study of Narayana Pandita's '*Gaṇitakaumudī (GK)*' chapters 13 and 14, Doctoral dissertation, Brown University 1993. Cf: *Historia Mathematica* 25 (1998)Pp.1-21, See P.21 Ref.2

2. Ed. by K.S. Shukla, Akhil Bharatiya Sanskrit Parishad, Lucknow, 1970.

3. Cf. Vinyasagara, M., *Vṛtta Mauktika,* Rajasthan Oriental Research Institue, Jodhapur, 1965, p. 531.

Comparison of the colphon giving the name of the author as Nārāyaṇa Paṇḍita and that of his father as Nṛsiṁha at the end of Part I of *Bījagaṇitāvataṃsa,*[4] with the colphons occuring at the end of the various sections of the *Gaṇitakaumudī*[5] leaves little doubt that the author of the *Bījagaṇitāvataṃsa* was the same Nārāyaṇa Paṇḍita as the author of the *Gaṇitakaumudī* . The identity of the authors of the *Gaṇitakaumudī* and the *Bījagaṇitāvataṃsa* is also established by the fact that most of the verses (including commentaries thereon) found in the sections dealing with *kuṭṭaka* and *varga-prakṛti* in the two works are almost literally the same. There is also a reference in the *Gaṇitakaumudī* to the author's *Bījagaṇita*[6].

4 . इति सकलकलानिधिनरसिंहनन्दन–गणितविद्याचतुरानन–
नारायणपण्डितविरचिते बीजगणितावतंसे वर्गप्रकृतिः समाप्ता ।
-NBi. I, p. 44.

5.(i) इति सकलकलानिधि नरसिंहनन्दन गणितविद्याचतुरानन
श्री नारायणपण्डित विरचितायां गणितकौमुद्यां मिश्रव्यवहारः ॥
– *GK,* I, p.105.

(ii) इति श्री सकलकलानिधि नरसिंहनन्दन गणित विद्याचतुरानन नारायण पण्डित विरचितयां गणितपाट्यां कौमुद्याख्यायां क्षेत्रव्यवहारः समाप्तः ।
– *GK,* II, p.192.

(iii) इति सकलकलानिधि नरसिंहनन्दन गणितविद्याचतुरानन श्री नारायणपण्डित विरचितायां गणित पाट्यां कौमुद्याख्यायां कुट्टकोनाम नवमो व्यवहारः समाप्तः ॥ – *GK,* II, p.232.

(iv) इति श्री सकलकलानिधि नरसिंहनन्दन गणितविद्याचतुरानन नारायण पण्डित विरचितयां गणितपाट्यां कौमुद्याख्यायां वर्गप्रकृतिर्नाम दशमोध्यायः समाप्तः । – *GK,* II, p.245.

6 . *Gaṇitakaumudī* (*GK*) Part I, p.13, lines 15-17.

The verse occuring in the end of the *Gaṇitakaumudī (G.K. Part II pp.410)* shows that the author's father Narasiṁha (Skt. Nṛsiṁha) was a respectable brahmaṇa well versed in Śṛtis and Smṛtis and a devotee of God Śiva and that besides being a good scholar of the Śāstras , he had acquired a high degree of proficiency in the mechanical science, the science of arms (Śastras,) and logic.[7]

Nārāyaṇa Paṇḍita lived in the fourteenth century of the Christian era. The concluding verse of the *Gaṇitakaumudī* shows that the *Gaṇitakaumudī* was completed on Thursday, the second tithi of the dark fortnight of the month Kārtika, Saṃvastara Durmukha,[8] śaka year 1278. This corresponds to Thursday, October 14, A.D.1356. The *Bījagaṇitāvataṃsa* which is mentioned

7. आसीत् सौजन्यदुग्धाम्बुधिरवनिसुरश्रेणिमुख्यो जगत्यां
प्रख्यः श्रीकण्ठपादद्वयनिहितमनाः शारदाया निवासः ।
श्रौतस्मार्तार्थवेत्ता सकलगुणनिधिः शिल्पविद्याप्रगल्भः
शास्त्रे शस्त्रे च तर्के प्रचुरतरगतिः श्रीनृसिंहो नृसिंहः ॥१॥
–*GK,* II, p. 410.

8. गजनगरविमित १२७८ शाके दुर्मुखवर्षेच वाहुले मासि ।
धातृतिथौ कृष्णदले गुरौ समाप्तिगतं गणितम् ॥ ५ ॥
–*GK*, II, p. 411.

इति श्री सकलकलानिधि श्रीमन्नृसिंह नन्दन गणितविद्याचतुरानन नारायण पण्डित विरचितयां गणितपाट्यां कौमुद्याख्यायां भद्रगणितंनाम चतुर्दशो व्यवहारः । समाप्तेयं गणितकौमुदी । –*GK*, II, 412

in the *Gaṇitakaumudī* [9] was evidently written prior to this date.

It is not known where the author lived and worked, but the distribution of manuscripts indicates that he was possibly from North India.

***Bījagaṇitāvataṃsa*:**

The name *Bījagaṇitāvataṃsa* is a compound formed by the words *Bījagaṇita,* meaning "algebra" (lit. "the science of calculation with elements"), and a*vataṃsa,* meaning " a garland or any ring shaped ornament", or "the crown". *Bījagaṇitāvataṃsa* thus means "a garland of the elements of algebra" or "The Crown Of Algebra".

The work availabale now, in printed form, published by Akhila Bharatiya Sanskrit Parishad, Lucknow, in 1970, is incomplete. It has been mentioned by the editor, Kripa Shankar Shukla, that the crtical edition brought out is based on an incomplete manuscript containing the whole of Part I and a few opening lines of part II, which was acquired by the late Dr. A. N. singh, belongs to their collection.

Like other Hindu works on algebra, it is divided into two parts, Part I dealing with algebraic processes essential in solving algebraic equations (bījopayogi-*gaṇita*) and Part II dealing with algebraic equations (bīja).

Part I sets forth the following three topics :

[9] . अत्र पाटीगणिते खहरे कृते लोकस्य व्यवहृतौ प्रतीतिर्नास्तीत्यतोऽत्र खहरो नोक्तः । अस्मदीये बीजगणिते बीजोपयोगित्वात् तत्र खहरः कथितः । -*Gaṇitakaumudī* (GK) Part I, p.13, lines 15-17.

(1) Algebraic operations (of addition, subtraction, multiplication, division, squaring and extracting the square-root) for each of the following :

(i) Positive and negative numbers (*dhanarṇa*),
(ii) The zero (*śūnya*),
(iii) Single unknown (*avyakta*),
(iv) More unknowns characterized by colours (*varṇa*),
(v). surds (*karaṇī*).

Thus, operations in all works out to 30 only. But the number of operations as mentiond in the work are thirty-six (*ṣaṭtriṃśat -parikarmāṇi*).

(2) The Pulveriser (*kuṭṭaka*), i.e. analysis pertaining to the indeterminate equation $(ax \pm c)/b = y$.

(3) The Square-nature (*varga-prakṛti*), i. e., analysis pertaining to the indeterminate equation :

$$Nx^2 \pm c = y^2.$$

It is mentioned by the editor, that the manuscript breaks off after giving the list of contents of Part II and an example of a linear equation in one unknown.

From the opening lines of Part II of the work, we learn that it dealt as usual with the following algebraic equations (*bīja*):

(i) Linear equations in one unknown (*avyakta-samīkaraṇa*).
(ii) Linear equations in more than one unknown (*varṇa- samīkaraṇa*)
(iii) Elimination of the middle term (*madhyamā-haraṇa),* or the quadratic equation.

(iv) Equations involving the product of different unknowns (*bhāvita-samatva*).

The method for finding the square root of a surd of the type ($a + \sqrt{b} + \sqrt{c} + \sqrt{d}$) as given in Rule 46-49,[10] and the method of finding the approximate values of the square-root of a non square number (quadratic surd) as given in rule 86, were given by Nārāyaṇa Paṇḍita for the first time; and are note worthy.[11]

Special Notable Features In Nārāyaṇa Paṇḍita's *Gaṇitakaumudī* :

i) K.S. Shukla has shown that the so called Fermat's factorization method was given by Nārāyaṇa in the *Gaṇitakaumudī* [12] about three centuries before it struck the mind of the French mathematician Pierre de Fermat in 1643.

ii) Nārāyaṇa Paṇḍita was the first Indian to give rules for testing arithmetic operations by casting out any desired number [13].

iii) There are quite a few problems which were proposed and solved for the first time by Nārāyaṇa. For example, mention may be made of the following:

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10 . "SURDS IN HINDU MATHEMATICS" by Datta and Singh revised by K.S. Shukla IJHS. 28 (3), 1993 Pp. 254-264. see Pp. 263.

11 . 'APPROXIMATE VALUES OF SURDS IN HINDU MATHAMATICS' by Datta and Singh revised by K.S. Shukla *Indian Journal of History of Science (IJHS)* **28** (3), 1993. 265-275.

12 . *G.K.*II Ch.xi Vs 51/2-81/2 Pp. 245-247 K.S.Shukla : "Hindu Methods for finding factors or divisors of a number', Ganita 17, No.2 Pp. 109-117. 1966.

13 . *G.K.*II Pp. 255 Vs 11, *History of Hindu Mathematics (HHM) I.* Pp. 183-184.

"A cow gives birth to a she-calf every year; and the calves themselves start giving birth when three years old. 0 learned one, tell the number of progeny produced by the cow in 20 years"[14] (answer: 2745).

iv) The age old practice of performing area and perimeter preserving transformation led Indians to marvellous discovery of the third diagonal of a cyclic quadrilateral. Nārāyaṇa Paṇḍita dealt with the subject[15] and knew the beautiful relation: R = xyz / 4A between the diagonals x,y, z, area A, and circum radius R of any cyclic quadrilateral.[16]

v) He has determined all rational triangles whose sides are consecutive integers[17] *(G.K.II.* pp. 150-153; Vs-118-120). These are with side lengths: (3, 4, 5) ; (13, 14, 15) ; (51 ,52, 53) ; (193, 194, 195) ; (2701, 2702, 2703) ; (10083, 10084, 10085) ; (37633, 37634, 37635) etc. The areas of these triangles are 6, 84, 1170, 16296, 136 1340, 44031786 and (12266893267/2) square units respectively.

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14 . प्रतिवर्षं गौः सूते वर्षत्रितयेन तर्णकी तस्याः ।
विद्वन् विंशति वर्षैर्गोरेकस्याश्च सन्ततिं कथय ॥१६॥
-GK. Part-I. Ch.III. Ex.16. Pp.126-127

15 GK. Part-II Ch.IV – Vs-131-148. Pp.164-192.
Paramanand Singh: 'Quadrilateral and Its third diagonal' *Vishveshvaranand Indological Journal.* Vol.XXI. Part i-ii 1983, Pp 219-227.

16 . चतुराहतफलविहृते त्रिकर्णघातेऽथवा हृदयम् ।१३८(i)।
-*GK,*II, iv, 138(i), p. 174.

17. K.Venkatachaliyengar : "The Development of Mathematics in Ancient India", "Scientific Heritage of India" Proceedings of a National seminars, June, 1988 The Mythic Society, Bangalore.Pp.47

vi) Nārāyaṇa Paṇḍita has extended and generalized the concepts of pratyayas of Sanskrit prosody to combinatorics. He dealt with the subject in detail and has devoted a complete chapter viz. *Aṅkapāśa* of his work *Gaṇitakaumudī* for the purpose.

The treatment of complicated sequences and net of numbers *(aṅkapāśa)*[18] is note worthy. The rule for finding the general term of Sūcī-paṅkti ("Spindle line") is equivalent to finding the coefficient of a general term in multinomial expansion.

viii) Nārāyaṇa's method of summation of any order triangular number or Vārasaṅkalita shortly, has established a way for the development of integration series in connection with quadrature of the circle and allied problems.

18. For details, See The *Gaṇitakaumudī* of NārāyaṇaPaṇḍita chapter XIII, English translation with notes by Paramanand Singh, *Gaṇita Bhāratī* Vo. 23, No.1-4 (2001) 18-82.

Paramanand Singh: (i) 'The So-called Fibonacci Numbers in Ancient and Medieval India' *Historia Mathematica* Vol.XII (1985) Pp.229-244 and (ii) 'Binomial and Multinomial Theorems in India and Central Asia' ; *Interaction between Indian and central Asian Science and Technology in Medieval Times (Indo-Soviet Joint Monograph Series)* (Ref.28.1) Vol.I Pp.297-311 INSA. New Delhi (1990)

Heroor Venugopal. D. *Development Of Combinatorics From The Pratyayas In Sanskrit Prosody,*.(2011). Sanskrita Bharati, Jhandevalan, New Delhi 100 055.

Also see : Takanori Kusuba, Combinatorics and Magic squares in India, Doctoral Dissertation Brown University 1993.

viii)The last chapter of the work is on magic squares, whose treatement is comprehensive and touches new heights. Interestingly, magic squares are said to have been first taught by lord Shiva to Maṇibhadra, the king of Yakṣas, on acount of which, the *Gaṇita* is called after his name as *Bhadragaṇita (G.K*. II Intro pp.ii-iv).

In *Tantra* Śāstra they are called *yantras.* As they were supposed to possess mystical properties, they were kept secret and were not dealt with in Arithmetic by Indian mathematicians. But Nārāyaṇa, defying this superstitious belief, touched upon the subject and gave definite rules for the construction of them containing an odd or even cells.[19] Nārāyaṇa's Magic Square of constant 111 is shown below.

1	35	4	**33**	32	**6**
25	**11**	9	28	**8**	30
24	**14**	18	**16**	**17**	22
13	**23**	19	**21**	**20**	15
12	**26**	**27**	10	**29**	7
36	2	**34**	3	5	**31**

M= 111

19 . G.K. Part-II. Pp. XV-Xvi and Ch.XIV GKI SD. (Ref.15) Pp.118-122. R.C.Gupta: Yantras or Mystic Diagrams: A wide Area For Study in Ancient And Medieval Indian Mathematics. *IJHS.* Vol.42. No.2 (2007) Pp.163-204.

श्रीनारायणपण्डितविरचितः

बीजगणितावतंसः

एकमनेकस्योक्तं (नित्यं) व्यक्तस्य गुणवतो जगतः ।
गणनाविधेश्च बीजं ब्रह्म च गणितं च तद् वन्दे ॥१॥

"I adore that Brahma, also that science of calculation with unknown, which is the one invisible root-cause of the visible and multiple-qualitied universe, also of multitudes of rules of the science of calculation with the known."[1] ॥1॥

अजगोलोऽयमियानिति करकलितामलकसन्निभो येन ।
व्यक्तीचक्रे ह्यगणितगणितेन (च) तत् किमस्त्यन्यत् ॥२॥

"Is there any other thing (science) than the Infinite Mathematics by which one can manifest (measure) Brahmāṇḍa (egg of Brahman), (the source of all universe) so clearly like the emblic fruit placed in one's palm ? ॥ 2 ॥

गणितमिति नाम लोके ख्यातमभूदगणितस्य शास्त्रस्य ।
अगणितविक्रमविष्णोस्त्रिविक्रमश्चेति नामेव ॥३॥

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[1] . HHM, II, p.5.

"As Vishnu, of infinite valour, is named as *Trivikrama,* so also the infinite science of computation (Mathematics) is renowned as *Gaṇita* in this world." ॥ **3** ॥

सद्गुरुकृपयाऽनुभवैरभ्यासैः परमतत्त्वमिव योगी ।
यो वेत्ति कर्म साङ्ख्यं स भवति सङ्ख्यावतां धुर्यः ॥४॥

"As a Yogi, a Spiritual person, attains the Supreme Truth by the grace of his preceptor, experience and constant practice, a person by knowing different operations pertaining to numbers can become a leader among mathematicians." ॥ **4** ॥

यो यो यं यं प्रश्नं पृच्छति सम्यक्करणं न तस्यास्ति ।
व्यक्तेऽथाव्यक्ते तु प्रायस्तत्करणमस्त्येव ॥५॥

व्यक्तक्रियया ज्ञातुं प्रश्नानखिली भवन्ति नाल्पधियः ।
बीजक्रियां च तस्माद् वच्मि व्यक्तां सुबोधां च ॥६॥

"People ask questions whose solutions are not to be found by arithmetic ; but their solutions can be generally be found by algebra. Since less intelligent men do not succeed in solving questions by the rules of arithmetic, I shall speak of the lucid and easily intelligible rules of algebra."[2] ॥ **5-6** ॥

[2] . HHM, II, p. 5.

बीजोपयोगि–गणितम्

(१) षट्त्रिंशत् परिकर्माणि

(36 - Logistics)

(i). धनर्णषड्विधम

(6-OPERATIONS WITH POSITIVE AND NEGATIVE NUMBERS)

धनर्णं सङ्कलिते करणसूत्रमार्याद्वयम् –

रूपाणामव्यक्तानां नामाद्यक्षराणि लेख्यानि ।
उपलक्षणाय तेषामृणगानामूर्ध्वबिन्दूनि ॥७॥

1.1 : Rules for addition of positive and negative numbers :

" The initial letters of the names of knowns and unknowns should be written for implying them. Those (known and unknown) which are negative should be written with a dot (*bindu*) over them."[1] ॥ **7** ॥

योगे धनयोः क्षययोर्योगः स्यात् स्वर्णयोर्भवेद् विवरम् ।
अधिकादूननपास्य च शेषं तद्भावमुपयाति ॥८॥

"In the addition of two positive or two negative numbers the result is their sum ; but of a positive and a negative number, the result is their difference ; subtracting the smaller number from the greater, the remainder becomes of the same kind as the latter."[2] ॥ **8** ॥

[1] . HHM, II, p.17, fn.3, and p.14, fn.2.

[2] . (i) BrSpSi.xviii-30;(ii) GSS, i.50-51; (iii) Si.Se. xiv- 3 ; iii.-28;

उदाहरणम् –

रूपत्रयञ्च रूपकपञ्चकमस्वं धनात्मकं वाऽपि ।
वद सहितं झटिति सखे स्वर्णमृणं स्वं च यदि वेत्सि ॥१॥

Ex. 1 : **"**Oh friend, if you know, tell me at once, sum of : $\{(+3)+(+5)\}$; $\{(-3)+(-5)\}$; also $\{(3)+(-5)\}$; and $\{(-3)+(+5)\}$." ॥ **1** ॥

Solution : (i) $3 + 5 = 8$,
(ii) $-3 + (-5) = -8$,
(iii) $3 + (-5) = -2$,
(iv) $-3 + 5 = 2$.

इति धनर्णसङ्कलनम् ।

Thus ends the addition of positive and negative numbers.

धनर्णव्यवकलने सूत्रमार्यार्धम् –

स्वमृणत्वमृणं स्वत्वं शोधकराशेः समुक्तवद्योगः ॥ ९(i) ॥

1.2: Rule for subtraction of positive and negative numbers:

"Of the subtrahend affirmation becomes negation and negation affirmation ; then addition as described before."[3] ॥ **9(i)** ॥

उदाहरणम् –

रूपाष्टकं रूपकपञ्चकेन क्षयं क्षयेनापि धनं धनेन ।
धनं क्षयेण क्षयगं धनेन व्यस्तं च संशोध्य वदाशु शेषम् ॥२॥

(iv) BBi, R.3 ; Cf. HHM. II. p. 20. f.n. 4 and p.21. f.n. 1-5.,

[3] . (i). BrSpSi. Xviii.31-32; (ii).GSS.i.-51; (iii).Si.Se. xiv-3; (iv).BBi. R. ॥७॥ 5 ॥. Cf. HHM. II. p. 22.

EX. 2 : "(Subtract) $-5\,from-8$, $+5\,from+8$, and $+5\,from-8$, also after making the positive negative, and the negative positive and carrying out subtraction tell me the remainder." ॥ **2** ॥

Solution : (i) $-8-(-5) = -3$,
(ii) $8-(+5) = 3$,
(iii) $-8-(+5) = -13$,
(iv) $8-(-5) = 13$.

इति धनर्णव्यवकलना ।

Thus ends the subtraction of positive and negative numbers.

अथ धनर्णगुणने सूत्रमार्यार्धम् –

ऋणयोर्धनयोर्घाते स्वं स्यादृणधनहतावस्वम् ॥९(ii)॥

1.3 : Rule for multiplication of positive and negative numbers :

"The product of two negative or two positive (numbers) is positive; the product of negative and positive is negative.[4] ॥ **9(ii)** ॥

उदाहरणम् –

रूपद्वयं रूपकपञ्चकेन धनं धनेन क्षयगं क्षयेण ।
धनं धनेन क्षयगं धनेन निघ्नं पृथक् किं गुणने फलं स्यात् ॥३॥

Ex. 3 : "Tell me the products sevarally, in the multiplications : $(2\times 5), (-2\times -5), (2\times -5), and\ (-2\times 5)$." ॥ **3** ॥

Solution : (i). $2 \times 5 = 10$,
(ii). $-2\times -5 = 10$,

[4]. (i). BrSpSi. Xviii.-33; (ii).GSS.i.-50; (iii).Si.Se. xiv-4; (iv).BBi.R. ९/7; Cf. HHM. II. pp. 22-23.

(ii) शून्यषड्विधम्

शून्यसङ्कलितव्यवकलितयोः करणसूत्रम् –

स्वर्णं शून्येन युतं विवर्जितं वा तयैव तद् भवति ।
शून्यादपनीतं तत् स्वर्णं व्यत्यासमुपयाति ॥११॥

2.1 - 2.2 : Rule for addition and subtraction with zero :

"When zero is added to or subtracted fom a positive or a negative number the number remains unchanged; and thus results in the same number. If a positive or negative number is subtracted from zero its sign is reversed."[1] ॥ **11** ॥

That is, i. $a + 0 = a$

ii. $a - 0 = a$

iii. $0 - (+a) = -a$

iv. $0 - (-a) = +a$

where 'a' is either a positive or a negative number.

उदाहरणम् –

रूपपञ्चकमृणं धनं सखे खेन युक्तमथवा विवर्जितम् ।
शून्यतः पृथगपास्य तानि वा किं भवेद् गणक मे पृथग्वद ॥६॥

Ex. 6 : "Oh friend, zero is added to or subtracted from -5 *and* $+ 5$. Or from zero those are subtracted severally ; O mathematician, what happens tell me (the results) separately." ॥ **6** ॥

Solution :

I. $-5 + 0 = -5$,
II. $5 + 0 = 5$,
III. $0 - (-5) = 5$,
IV. $0 - 5 = -5$.

इति शून्यसंकलनव्यवकलने ।

Thus ends the operations of addition and subtraction with zero.

[1] . BBi. R. १६/12. Cf. HHM, I p.242.

शून्यगुणने सूत्रमार्यार्धम् –

खं राशिना विगुणितं खं स्या–
द्राशिः खगुणश्च खं भवति ।१२(i)।

2.3 : Rule for multiplication with zero in half a stanza :

"Zero multiplied by a number is zero ; and a number multiplied by zero is also zero." । **12(i)** ।

That is, (v). $0 \times a = 0$

(vi). $a \times 0 = 0$

where 'a' is either a positive or a negative number.

उदाहरणम् –

धनर्णभूतैस्त्रिभिरेव सङ्गुणं
खं किं फलं स्यात्कथयशु तन्मे ।
धनात्मकाश्चाप्यधनात्मकास्त्रयः
खसंगुणाश्चापिफलं प्रचक्ष्व ॥ क्षेपक ॥

Example : Zero is multiplied separately by + 3 and also by -3 , tell me quickly what is the product ? Also tell the product when + 3 and -3 are multiplied by zero. ॥ **Extra** ॥

Solution :

I. $0 \times 3 = 0$
II. $0 \times -3 = 0$
III. $3 \times 0 = 0$
IV. $-3 \times 0 = 0$.

इति शून्यगुणनाविधिः

Thus ends the operations of multiplication with zero

खगुणादौ सूत्रम् –

खं राशिना विभक्तं खं स्याद्राशिः खभाजितः खहरः ॥१२॥

2.4 : Rule for(multiplication etc.) division with zero :

"Zero divided by a number is zero ; a number divided by zero is *kha-har* (that number with zero as denominator)."[2] ॥ **12** ॥ That is,

(vii). $\frac{0}{a} = 0$

(viii) $\frac{a}{0} = khahara$

where 'a' is either a positive or a negative number.

शेषविधौ सति खगुणश्चिन्त्यः शून्ये गुणे खहारश्चेत् ।
पुनरेव तदाविकृतो राशिज्ञयोऽत्र मतिमद्भिः ॥१३॥

2.5 : *kha-guṇa, kha-har, avikṛtiḥ* :

"If any further operations impend, zero having become a multiplier (of a number), then the zero must be retained as a multiple of that number (*kha-guṇa*). If zero is a multiplier (*kha-guṇa*), and afterwards again become a divisor (*kha-har*), then the number must be understood to be unchanged (*avikṛta*) here, by the learned."[3]॥**12(ii)–13** ॥

That is ,

(ix). $(a \times 0) \div 0 = a$

where 'a' is either a positive or a negative number.

[2]. - L(ASS)- 45-46 ; Cf. HHM. I. p. 242

[3] . L(ASS)- 45-46 ; Cf. HHM. I. p. 242

उदाहरणम् –

धनात्मकैश्चाप्यधनात्मकैस्त्रिभि–
विभाजितं खं फलमाशु मे वद ।
धनात्मकाश्चाप्यधनात्म (का) स्त्रयः
खभाजितास्त्वं गणक प्रवेत्सि चेत् ॥७॥

Ex. 7 : "Zero is divided by $+3$ and also by -3 ; tell me the quotients quickly. $+3$ and also by -3 are (severally) divided by zero, Oh mathematecian if you know, tell (the results)." ॥ **7** ॥

Solution :

I. $0 \div 3 = 0$
II. $0 \div -3 = 0$
III. $3 \div 0 = khahara = \infty$
IV. $-3 \div 0 = khahara = \infty$.

अत्र खहरगुण उच्यते –

शून्याभ्यासवशात्खतामुपगतो राशिः पुनः खोद्धृतो
व्यावृत्तिं पुनरेव तन्मयतया न प्रकृतिं गच्छति ।

आत्माभ्यासवशादनन्तममलं चिद्रूपमानन्ददं
प्राप्य ब्रह्मपदं न संसृतिपथं योगी गरीयानिव ॥१४॥

2.5.1 : Property of *kha-guṇa* :

"Once a quantity becomes zero through multiplication by zero, and then again if divided by zero, it can never be reverted to its original form (or value) "just as" a yogin will not revert to rebirth after attaining the blissful state of Brahma."[4] ॥ **14** ॥

[4] . R. C. Gupta : Zero in the mathematical system of India' in "*The concept of Śūnya*" New Delhi, 2003.

प्राक्तनश्लोकश्च –

अस्मिन् विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु ।
बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत् ॥१५॥

2.5.2 : Rule stated by former scholar (Ancient-rule)[5] :

Property of *kha-hara:*

"In this quantity consisting of that which has zero (or cipher) for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth." ॥ **15** ॥

इतिभागहारः ।

Thus ends the operation of division.

खयोग–वियोग–गुणन–भजन–वर्ग–वर्गमूलेषु सूत्रम् –

खं खयुतं रहितं वा खं स्यात् खेनाहतं च विहृतं वा ।
खस्य कृतिः खं खपदं खमेव सर्वत्र विज्ञेयम्[६] ॥१६॥

2.6: Rule for addition, subtraction, multiplication, division, evolution and involution of zero :

"When zero is added to or subtracted from zero the result is zero ; zero multiplied by zero, or zero divided by zero, is zero. square of zero is zero; square-root of zero is also zero; thus everywhere (i.e. in each operation) the result is to be understood as only zero." ॥ **16** ॥ That is,

---

[5] . This is the last verse in the section of operations with zero, in the *Bījagaṇita* of Bhāskara II. [BBi. R. २०/16.] Cf. HHM. I. p. 243.

[6] Serial number of this verse is mentioned as 15 inthe printed edition.

i. $0 + 0 = 0$
ii. $0 - 0 = 0$
iii. $0 \times 0 = 0$
iv. $0 \div 0 = 0$
v. $0^2 = 0$
vi. $\sqrt{0} = 0.$

उदाहरणम् –

खे शून्येन युते च किं विरहिते किं खेन निघ्ने च किं
किं भक्ते किमु वर्गिते कथय भो मूलीकृते किं सखे ।
राशिः कोऽपि खसङ्गुणो निजदलेनाढ्यः खसंभाजितो
जाता द्वादश तं द्रुतं वद दृढां प्रौढिं प्रयातोऽसि चेत् ॥८॥

Ex. 8 : (i). Find : $0 + 0 = ?$; $0 - 0 = ?$; $0 \times 0 = ?$; $0 \div 0 = ?$; $0^2 = ?$; $\sqrt{0} = ?$

(ii). $\left\{(x \times 0) + \frac{1}{2}(x \times 0)\right\} \div 0 = 12$, solve for x .

Solution : Following Nārāyaṇa Paṇḍita,

step-1 : $\left\{(x \times 0) + \frac{1}{2}(x \times 0)\right\} = 0 \times \left(x + \frac{1}{2}x\right)$

step-2 : $0 \times \left(\frac{3}{2}x\right) \div 0 = \frac{3}{2}x$

step-3 : $\frac{3}{2}x = 12 \qquad \therefore x = \frac{2 \times 12}{3} = 8.$

Here, assuming arbitrarily the number 2, for the unknown, Nārāyaṇa obtains the valueof LHS as 3. Then by the rule of three he arrives at the result : 8. In the course of solution of the example, Nārāyaṇa remarks (p.7) :

शून्यं गुणकः शून्यं भागाहारोऽतो गुणन भजने न कार्ये, ...।
यः कश्चिद् राशिः केनचिद् गुणितः पुनस्तेनैव भक्तश्चेदविकृत एव न भवति तर्हि गुणनभजने वृथा ।

इति शून्यस्य षड्विधम् ।

Thus ends the six operations with zero

2.7 : Infinity :

Bhāskara II makes it clear that infinity or infinite quantity (*ananta rāśiḥ*) is called *khahara.* He was aware of the fact that the *khahara* remains unchanged by addition or subtraction of any finite number *N.*

That is,

$$khahara \pm N = khahara \quad \ldots\ldots(\mathrm{x})$$

or symbolically: $(\pm)\infty \pm N = (\pm)\infty$.

In this connection the exposition of commentator Gaṇeśa (son of Keśava Daivjña) is quite relevant. In his *Buddhivilāsinī* commentary on *Līlāvatī* (A.D. 1545) Gaṇeśa Daivjña remarks (p.39) that :

खं हरो यस्येति खहरः । अस्य राशेरियत्ता कर्तुं न शक्यते । यतोऽसौ राशिः केनचिद्युक्तो हीनो वाऽविकृतः स्यात् । तथा हि – समच्छेदकरणेऽन्योन्यहाराभिघाते नान्यो राशिः शून्यमेव स्यात् ।

"A quantity the denominator of which is zero is *khahara;* i.e., $\frac{a}{0}$. It is an indefinite and unlimited or infinite quantity: since it cannot be determined how great it is. It is unaltered by the addition or subtraction of finite quantities: since in preliminary operation of reducing both fractional expressions to a common denominator, preparatory to taking their sum or difference, both numerator and denominator of the finite quantity vanish.[7] "

About the evolution of *khahara* $(a/0)$ [from the fraction (a/b)] and its nature, (by numerical consideration only), Kṛṣṇa (son of Ballāla) who wrote the commentary

[7] . HHM. I. p. 244.

Navāṅkura (also called *Bījapallava* etc.,) on *Bījagaṇita* of Bhāskara II in about 1600 A. D. remarks (p.28) :

यथा यथा भाजकापचयस्तथा तथा लब्धेरुपचयः तथा सति भाजके परमापचिते लब्धेः परमोपचयेन भाव्यम् । लब्धेश्चेदियत्तोच्येते तर्हि परमत्वं न स्यात्ततोऽप्याधिक्यसंभवात् । अतो लब्धेरियत्ताभाव एव परमत्वं तदेवमुपपन्नं खहरो राशिरनन्त इति ।

"As much as the divisor (b) is diminished, so much is the quotient (a/b) increased. If the divisor is reduced to the utmost (*parama*), the quotient is increased to the utmost. But, if it can be specified, that the amount of the quotient is so much, it has not been raised to the utmost because a greater than that is possible. Therefore the quotient is great without limit, and the resulting *khahara* quantity is infinite (*ananta*).[8] "

We may say that he was aware of the limit

$$\lim_{\epsilon \to 0}\left(\frac{N}{\epsilon}\right) = \infty \text{ (numerically) } \ldots\ldots(\mathrm{xi})$$

Earlier some misunderstanding and confusion prevailed about Bhāskara's statement , that the *khahara* remains unchanged by addition or subtraction of finite quantity. For instance, Gaṇeśa (son of Ḍhunḍhirāja), the author of *Gaṇitamañjarī* (circa 1560 A.D.), combined $(11/2)$ to the *khahara* $(4/0)$ to get

$$\frac{4}{0} \pm \frac{11}{2} = \frac{4\times 2 \pm 11\times 0}{0\times 2} = \frac{8}{0}$$

by the usual arithmetical process. Thus the *khahara* $(4/0)$ has changed to *khahara* $(8/0)$ and so Gaṇeśa concluded that there is change (*vikāraḥ*) in the *khahara* contrary to

[8] . HHM. I. p. 244.

what Bhāskara II believed (apparently the confusion is due to lack of a single form of a symbol for *khahara*).

However, Kṛṣṇa refutes the above criticism by saying that although the forms (such as $4/0$ and $8/0$) are different, their value is same (each being infinity). In other words he showed that (pp.29-30) :

$$\frac{a}{0} = \frac{b}{0} \quad .... \quad .... \quad .... (xii)$$

For this he takes an astronomical problem. We know that the length of the shadow s of a gnomon of length g when the sun's altitude is α is given (in ancient form) by:

$$s = g.\left(\frac{R\cos\alpha}{R\sin\alpha}\right) \quad .... \quad .... (xiii)$$

where R is the radius of the circle of reference (chosen arbitrarily). At the sun-rise, the altitude is zero and the shadow will be given by

$$s = \frac{g.R}{0} \quad ..... \quad ... (xiv)$$

The usual value of g was 12 *aṅgulas*. For a few values of *R*, Kṛṣṇa correctly gets the corresponding values of s as follows :

R =	3438	120	100	90
s =	41256/0	1440/0	1200/0	1080/0

Since even with the choice of different radii (*nānātrijyā*), the length of the shadow of a gnomon at any time must be same (*chāyā tulyaiva*), he rightly concludes that the above various values of the shadow must be equal. This illustrates the truth of (*xii*). Thus each form of above present the same *khahara* (here infinite shadow)[9].

[9]. R. C. Gupta : Zero in the mathematical system of India' in *The concept of Śūnya* New Delhi, 2003.

2.8 : Zero as an Infinitesimal :

The idea of zero as an infinitesimal is quiet evident from the term *kha-guṇa* used while stating $a \times 0 = 0$,and then $\frac{a\times 0}{0} = a$ above. Kṛṣṇa in his commentary, *Navāṅkura* (*ASS*. p.18) on *Bījagaṇita* of Bhāskara II proves the result, $0 \times a = a \times 0$ as follows :

गुण्यस्यापचयवशाद्गुणनफलस्यापचय इति तावत् प्रसिद्धम् ।...गुण्यस्य परमापचये गुणनफलस्यापि परमापचयेन भाव्यम् । परमापचये च शून्यतैव पर्यवस्यतीति शून्ये गुण्ये गुणनफलं शून्यमेवेति सिद्धम् । एवं गुणकापचयवशादपि गुणनफलेऽपचयात् गुणकस्यापि शून्यत्वे गुणनफलशून्यमेवेति सिद्धम् ।

" The more the multiplicand (*guṇya*) is diminished, the smaller is the product; and, if it be reduced in the utmost degree, the product is so likewise : now the utmost diminution of a quantity is the same with the reduction of it to nothing ; therefore, if the multiplicand (*guṇya*) be nought (i.e., zero), the product is zero (or cipher). In like manner, as the multiplier (*guṇaka*) decreases, so does the product; and, if the multiplier (*guṇaka*) be zero (or nought), the product is so too.[10] "

In the above, zero is conceived of as the limit of a diminishing quantity.

[10] . . HHM. I. p. 243.

2.9 : Indeterminate Forms :

In connection with the unusual rule, $\frac{a\times 0}{0} = a$ the exposition of commentator Gaṇeśa (A.D. 1545) in his *Buddhivilāsinī* commentary on *Līlāvatī* (p.40) is as follows:

राशेः शून्ये गुणके प्राप्ते तस्यान्यो विधिश्चेदस्ति तदा खगुणो राशिः खं स्यादिति न कार्यम् । किंतु शून्यं तत्पार्श्वे गुणकस्थाने स्थाप्यमिति । ततः शेषविधानं कृते पुनः खंहरश्चेत्तदा तयोः शून्यगुणकहरयोस्तुल्यत्वेन नाशः कार्यः ।

"When a quantity has zero multiplier and other operation is there, then the rule $a \times 0 = 0$ should not be applied. But the zero should be placed by its side as a multiplier. If for the remaining operation , zero is a divisor, then due to equal multiplier and divisor, the zero (in the numerator and denominator) should be cancelled (*nāsah kāryaḥ*)."

That is, we get (viii) by cancelling the zero in the numerator of the left hand side with the zero of the denominator.

Bhāskara's own example[11] is :

कः खगुणो निजार्धयुक्तस्त्रिभिश्च गुणितः खहृतस्त्रिषष्टिः ॥

That is to find x from the equation

$$\left[\frac{\{(x\times 0)+\frac{1}{2}(x\times 0)\}\times 3}{0}\right] = 63.$$

This can be written as : $\left(\frac{3}{2}\right)\cdot(x\times 0)\cdot\left(\frac{3}{0}\right) = 63$

[11] . L(ASS). p .40. vs. 47.

or $$\left(\frac{9}{2}\right)\cdot\left(\frac{x\times 0}{0}\right) = 63.$$

Now by applying $\frac{a\times 0}{0} = a$ *we easily get* the solution $x = 14$ as given by Bhāskara II.

His other examples[12] are:

$$\left\{\left(\frac{x}{0}+x-9\right)^2+\left(\frac{x}{0}+x-9\right)\right\}0 = 90$$

giving $x = 9; and$

$$\left\{\left(x+\frac{x}{2}\right)\times 0\right\}^2 + 2\left\{\left(x+\frac{x}{2}\right)\times 0\right\}\div 0 = 15$$

giving $x = 2.$

The answers of this and the previous example are incorrect because 0^2 has been taken to be equal to 0.

Of course, we know that the cancellation rule :

$$\frac{(x\times n)}{n} = x$$

is not valid if n is zero (or infinity). But what to say of ancient and medieval writers, even modern scholars have a pitfall in this respect.

Actually, the value of the left-hand side of $\frac{a\times 0}{0} = a$ is indeterminate because $a\times 0$ is zero. In fact, once a quantity x becomes zero through multiplication by zero, it can never be reverted to its original form (or value) "just as", in the words of Nārāyaṇa Paṇḍita (1356) stated above: "A yogin will not revert to rebirth after attaining the blissful state of Brahma." [N.Bi. I. R-14]

[12] See examples 69 and 70 stated in verses 135 and 136 respectively, in the *Bījagaṇita* of Bhāskara II .

(iii) अव्यक्तषड्विधम्

अथाव्यक्तसङ्कलनव्यवकलने करणसूत्रम् –

यावत्तावत्–कालक–नीलक–पीताश्च लोहितो हरितः ।
श्वेतक–चित्रक–कपिलक–पाटलकाः पाण्डु–धूम्र–शबलाश्च ॥१७॥

श्यामलक–मेचक–धवलक–पिशङ्ग–शारङ्ग–बभ्रु–गौराद्याः ।
गणनोत्पत्त्यै विहिता संज्ञाश्चाव्यक्तमानानाम् ॥१८॥

(Notations for unknowns) :

" *Yāvat-tāvat* (so much as), *kālaka* (black), *nīlaka* (blue), (*pītaka*) (yellow), *lohitaka* (red), *harītak* (green), *śvetaka* (white), *citraka* (variegated), *kapilaka* (tawny), *pātalaka* (pink), *pāṇḍu* (pale-white, or yellowish-white), *dhūmraka* (smoke-coloured), *śavalaka* (spotted), *śyāmalaka* (blackish), *mecaka* (dark blue), *dhavalaka* (white), *piśaṅga* (reddish-brown), *śāraṅga* (green), *babhru* (deep-brown, reddish-brown), *gauraḥ* (white, yellowish, pale-red), etc. colours have been prescribed as the notations for the measures of the unknowns, for the purpose of calculating with them.[1] ॥ **17-18** ॥

Nārāyaṇa has further added (in the commentary) :

अथवा वर्णाः कादयः , अथवा मधुरादयो रसपर्यायाः, अथवा असदृशप्रथमाक्षरनामपदार्थाः कल्प्यन्ते ।

"Or the letters of alphabets (*varṇa)* such as *ka,* etc., or the series of flavours such as *madhura* (sweet), etc., or the names of dissimilar things with unlike initial letters, are assumed (to represent the unknowns)."

[1] . BBi. R. २१/17; Cf. HHM, II, pp.18-19.

3.1-3.2: Rules for addition and subtraction of unknowns :

वर्णेषु च समजात्योर्योगः कार्यस्तथा वियोगश्च ।
असदृशजात्योर्योगे पृथक्स्थितिः स्याद् वियोगे च ॥१९॥

" Of the colours or letters of alphabets (representing the unknowns) addition should be made of those which are of the same species ; and similarly subtraction. In the addition and subtraction of those of different species the result will be their putting down severally."[2] ॥ **19** ॥

क्षयधनयोर्युतिवियुती गुणनभजने वर्गवर्गमूले च ।
अव्यक्तानां बहूनां रूपवदुपलक्षणं भवति ॥२०॥

"In the addition, subtraction, multiplication, division, squaring, and extracting the square-root of two or more unknowns with positive or negative signs, the method will be the same as explained in arithmetic."॥ **20** ॥

उदाहरणम् –

अव्यक्तषट्कं च धनं सरूपमव्यक्तयुग्मञ्च विपञ्चरूपम् ।
किमेतयोरैक्यमृणं धनञ्च तद्व्यस्तयोः सङ्कलन वदाशु ॥९॥

Ex. 9 : "What is the sum of the two quantities : $(6x + 1)$ and $(2x - 5)$ with both positive and negative signs? Also tell the sum after reversing the sign." ॥ **9** ॥

Solution : In accordance with the rule stated in verse 19, adding the unknowns of the same species and the absolute numbers mentioned in the example :

$$\begin{array}{cccc} 6x + 1 & -6x - 1 & 6x + 1 & -6x - 1 \\ \underline{2x - 5} & \underline{2x - 5} & \underline{-2x + 5} & \underline{-2x + 5} \\ 8x - 4 & -4x - 6 & 4x + 6 & -8x + 4 \end{array}$$

[2] . (i). BrSpSi.xviii-41; (ii). BBi. । २१। 18 ।. Cf. HHM. II. pp. 25-26.

उदाहरणम् –

अव्यक्तवर्गद्वितयं सरूप–
मव्यक्तयुग्मेन युतं च किं स्यात् ।
अव्यक्तषट्कं क्षयगं सरूपं
शोध्यं तु षड्रूपसुसंयुतेभ्यः ॥
अव्यक्तकेभ्यो गणक ! प्रचक्ष्वा–
ष्टाभ्योऽवशेषं यदि वेत्सि बीजम् ॥१०॥

Ex. 10 : "What is the sum of $(2x^2 + 0.x + 1)$ and $(2x)$? If you know algebra tell me the remainder when $(-6x + 1)$ is subtracted from $(8x + 6)$." ॥ **10** ॥

Solution :

$$\begin{array}{r} 2x^2 + 0x + 1 \\ \underline{0x^2 + 2x + 0} \\ 2x^2 + 2x + 1 \end{array} \qquad \begin{array}{r} 8x + 6 \\ \underline{(-)\ -6x + 1} \\ 14x + 5 \end{array}$$

इत्यव्यक्तसंकलनव्यवकलने ।

Thus ends the operations of multiplication and divisions with the unknowns

अथाव्यक्तगुणने करणसूत्रम् –

स्याद्रूपवर्णघाते वर्णो, द्वित्र्यादितुल्यजातिबधे ।
तत्कृतिघनादयः स्युः, तद् भावितमसमजातिवधे ॥२१॥

गुणकारसमुत्थानि स्वजातिखण्डानि योजयेदेव ।
अव्यक्तवर्गकरणीगुणनासु व्यक्तवज्ज्ञेयम् ॥२२॥

3.3 and 3.5 : Rule for multiplication (and squaring) of unknowns :

"A known quantity multiplied by an unknown becomes unknown; the product of two, three or more unknowns of like species is its square, cube etc.; and the product of those of unlike speices is their *bhāvita*.[3]

The multiplicand is put down separately in as many places as there are terms in the multiplier and is then (severally multiplied by those terms; the products are then) added together as in the case of knowns; Here, in the squaring and multiplication of unknowns, should be followed the method of multiplication by component parts, as explained in arithmetic.[4] ॥ **21-22** ॥

उदाहरणम् –

यवत्तावद् द्वितयसहितं रूपषट्कं विनिघ्नं
यावत्तावत्त्रितयरहितै रूपकैः पञ्चभिश्च ।
गौणं किं स्याद् वद मम फलं हे सखे ! कल्पयित्वा
व्यक्ते वर्णे पटुरसि यदि त्वं गुणाकारमार्गे ॥११॥

Ex. 11 : "Oh, friend ! what is the product when $(2x + 6)$ is multiplied by $(-3x + 5)$? Also tell me the product after reversing the sign of the multiplicand and the multiplier, if you are skilful in multiplication." ॥ **11** ॥

Solution :

EX.11.1 :

multilpicand	$2x + 6$
multiplier	$\underline{-3x + 5}$

---

[3] Cf. BrSpSi.xviii.-42. BBi. R. २६/21.

[4] . HHM II. p.26, f.n.4 and 5

Put down the multiplicand at two places as there are two terms in the multiplier; and then multiply the multiplicand by each term of the multiplier separately.

$$\begin{array}{cc} (2x+6) & (2x+6) \\ \underline{\times\ (-3x)} & \underline{\times\ (+5)} \\ (-6x^2-18x) & (10x+30) \end{array}$$

Sum of the products is the desired result.

$\therefore$ product $(6x^2-18x)+(10x+30)$

or $-6x^2-8x+30.$

multilpicand	$2x+6$
Ex.11.2 : multiplier	$\underline{3x-5}$
product	$(6x^2+18x)+(-10x-30)$ or $6x^2+8x-30$

Ex.11.3 :

multilpicand	$-2x-6$
multiplier	$\underline{3x-5}$
product	$(-6x^2-18x)+(10x+30)$ or $-6x^2-8x+30$

अव्यक्तभागहारे सूत्रम् –

शुद्ध्यति यर्यैर्वर्णै रूपैर्भाजको हतो भाज्यात् ।
क्रमशः स्वेषु स्वेषु स्थानेषु फलानि तानि स्युः ॥२३॥

3. 4 : Rule for division of unknowns :

"By whatever unknowns and knowns, the divisor is multiplied (severally) and subtracted from the dividend successively so that no residue is left, they constitue the quotients at the successive stages."[5] ॥ **23** ॥

इत्यव्यक्तभागहारः ।

Thus ends the operation of division with the unknowns.

(अव्यक्तवर्गे उदाहरणम् –)

अव्यक्तानां रूपपञ्चोनितानां
षण्णां वर्गं वा युतानां प्रचक्ष्व ।
चेद् बीजज्ञोऽसि त्वमस्याः कृतेश्च
मूलं विद्वन् ! ब्रूहि तन्मे पृथक् किम् ॥१२॥

Ex. 12 : "Oh learned man ! If you know algebra, Tell me the square of $(6x-5)$ or $(6x+5)$ and also the square-root of the result, separately." ॥ **12** ॥

Solution :

(i). $(6x-5)^2 = (6x-5)\times(6x-5)$
$= (36x^2-30x)+(-30x+25)$
$= 36x^2-60x+25.$

(ii). $(6x+5)^2 = (6x+5)\times(6x+5)$
$= (36x^2+30x)+(30x+25)$
$= 36x^2+60x+25.$

5 . BBi. vs. ॥२१॥ 24 ॥ , HHM II. p.27, f.n.2 and 3

(अव्यक्त वर्गमूले करण)सूत्रम् –

मूलान्यादायादौ वर्गेभ्यस्तद् द्वयोर्घातम् ।
द्विगुणं जह्याच्छेषान् मूलमितीदं वदन्तीह ॥२४॥

3.6 : Rule for square-root of unknowns :

"First find the root of the square terms (of the given expression); then the product of two and two of them (roots) multiplied by two should be subtracted from the remaining terms; (the result thus obtained) is said to be the square-root here (in algebra)."[6] ॥ **24** ॥

Example : Find the square root of :

(i). $36x^2 - 60x + 25$;

(ii). $36x^2 + 60x + 25$

Solution :

$36x^2 - 60x + 25 = (6x)^2 + (-5)^2 + (6x \times -5) + (6x \times -5)$

∴ In accordance with the rule stated in verse 24,

$$\sqrt{36x^2 + 60x + 25} = 6x - 5.$$

Similaraly,

$36x^2 + 60x + 25 = (6x)^2 + (5)^2 + (6x \times 5) + (6x \times 5)$

∴ In accordance with the rule stated in verse 24,

$\sqrt{36x^2 + 60x + 25} = 6x + 5.$

इत्यव्यक्तवर्गमूले ।

Thus ends the operation of extracting the square-root of the unknowns.

इत्यव्यक्तषड्विधम्

Thus ends the six-operations with the unknowns.

[6] . BBi. ॥३१॥ 26 ॥ ; Cf. HHM II. p.28, f.n.1

(iv) वर्णषड्विधम्

उदाहरणम् –

यावत्तावत्त्रितयमधनं कालकाः षट् क्षयं भोः
विद्वन् ! नीलाष्टकमपि धनं पीतकाः स्वं च पञ्च ।
रूपाढ्यैस्तैर्द्विगुणितमितैस्तेऽपि युक्ता वियुक्ताः
जानासि त्वं यदि झटिति मे ब्रूहि वर्णाः कति स्युः ॥१३॥

तैस्तैर्हता कथय किं गुणने फलं स्याद्
भक्तं च तद् गणकवर्य गुणेन तेन ।
गुण्यस्य मे कथय वर्गमतश्च मूल
चेद् वर्णषड्विधविधाववधिं गतोऽसि ॥१४॥

Ex. 13 - 14 : "Oh scholar! The unknown quantity $(-3x - 6y + 8z + 5u + 1)$ is added to and subtracted from the twice of it ; if you know the operations of addition and subtraction, tell me quickly what is the sum and difference of them ?

O learned mathematician, tell me what will be the result in the multiplication of those quantities, and also in the division of that (product-obtained) by the multiplicand. Also tell me, if you have gone through the six operations with the unknowns, the square of the multiplicand and the root of that square." ॥ **14** ॥

Statement (*Nyāsa*): $(-3x - 6y + 8z + 5u + 1)$ and
$2 \times (-3x - 6y + 8z + 5u + 1) = -6x - 12y + 16z + 10u + 2.$

Solution :

Ex.13 (i).

$$\begin{array}{r} -6x - 12y + 16z + 10u + 2 \\ \underline{(+)\ -3x - 6y + 8z + 5u + 1} \\ -9x - 18y + 24z + 15u + 3 \end{array}$$

(ii)

$$\begin{array}{r} -6x - 12y + 16z + 10u + 2 \\ \underline{(-)\ -3x - 6y + 8z + 5u + 1} \\ -3x - 6y + 8z + 5u + 1 \end{array}$$

(iii)

$$\begin{array}{r} -3x - 6y + 8z + 5u + 1 \\ \underline{(-)\ -6x - 12y + 16z + 10u + 2} \\ 3x + 6y - 8z - 5u - 1. \end{array}$$

इति वर्णसङ्कलनव्यवकलने ।

Thus ends the operations of addition and subtraction with the unknowns.

Ex. 14(i)

multiplicand : $-3x - 6y + 8z + 5u + 1$

multiplier : $-6x - 12y + 16z + 10u + 2.$

$(-3x - 6y + 8z + 5u + 1) \times -6x = 18x^2 + 36xy - 48xz - 30xu - 6x$

$(-3x - 6y + 8z + 5u + 1) \times -12y = 72y^2 + 36xy - 96yz - 6oyu - 12y$

$(-3x - 6y + 8z + 5u + 1) \times +16z = 128\,z^2 - 48xz - 96yz + 80zu + 16z$

$(-3x - 6y + 8z + 5u + 1) \times 10u = 50\,u^2 - 30xu - 60yu + 80zu + 10u$

$(-3x - 6y + 8z + 5u + 1) \times 2 = \underline{-6x - 12y + 16z + 10u + 2}$

product : $18x^2 + 72y^2 + 128z^2 + 50u^2 + 72xy - 96xz - 60xu - 192yz - 120yu + 160zu - 12x - 24y + 32z + 20u + 2.$

(ii) dividend : $(18x^2 + 72y^2 + 128z^2 + 50u^2 + 72xy - 96xz - 60xu - 192yz + 160zu - 12x - 24y + 32z + 20u + 2).$

divisor : $(-3x - 6y + 8z + 5u + 1)$

quotient : $-6x - 12y + 16z + 10u + 2.$

इति वर्णगुणनभजने ।

Thus ends the operations of multiplication and division with the unknowns.

$(-3x - 6y + 8z + 5u + 1) \times -3x = 9x^2 + 18xy - 24xz - 15xu - 3x$

$(-3x - 6y + 8z + 5u + 1) \times -6y = 36y^2 + 18xy - 48yz - 30yu - 6y$

$(-3x - 6y + 8z + 5u + 1) \times +8z = 64\,z^2 - 24xz - 96yz + 40zu + 8z$

$(-3x - 6y + 8z + 5u + 1) \times +5u = 25\,u^2 - 15xu - 30yu + 40zu + 5u$

$(-3x - 6y + 8z + 5u + 1) \times 1 = \underline{-3x - 6y + 8z + 5u + 1}$

square : $9x^2 + 36y^2 + 64z^2 + 25u^2 + 36xy - 48xz - 30xu - 96yz - 60yu + 80zu - 6x - 12y + 16z + 10u + 1.$

(iii) Square-root of $9x^2 + 36y^2 + 64z^2 + 25u^2 + 36xy - 48xz - 30xu - 96yz - 60yu + 80zu - 6x - 12y + 16z + 10u + 1.$ is $(-3x - 6y + 8z + 5u + 1).$

इति वर्णवर्ग–वर्गमूले ।

Thus ends the operations of squaring and extracting square-root of the unknowns.

इति वर्णषड्विधम् ।

Thus ends the (six) operations with the unknowns.

(v) अथ करणीषड्विधम् ।

SIX OPERATIONS ON SURDS[1].

5.0 : करणी Karaṇī (Surd) :

The Sanskrit word *karaṇī* means "producer" , "that which makes". It seems to have been originally employed to denote the cord used for measuring (the side of) a square in the *Śulba Sūtras*. From that it came to denote the sides of a rectilinear figure of any shape, and then more particularly, the side of a square *(caturasra-karaṇī)*. Hence it came to denote the square-root of any number.

As late as the second century of the Christian era, Umāsvāti (c. 150) treated the terms *mūla* ("root") and *karaṇī* as synonymous. In later times the term is, however, reserved for a surd, *i.e.,* a square-root which can not be evaluated, but which may be represented by a line.

Nemicandra (c. 975) has occasionally used the generic term *mūla* to signify a surd.e.g., *daśa-mūla* = $\sqrt{10}$

The term *karaṇī* was however changed to *karṇa* meaning 'ear' which may be classified as 'bad-ear' and 'good-ear'. Bad-ear means 'inaudible', i.e. where root can

1. Datta B. B. and Singh A.N (Revised by Kripashankar Shukla) : 'SURDS IN HINDU MATHAMATICS' : *Indian Journal of History of Science (IJHS)* **28** (3), 1993. 254-264.

not be exactly determined and 'audible' whose root can be determined.[2]

The idea of terms 'audible' for rational number and 'inaudible' for irrational number used by the Arabian scholar al-Khawarizmi (c.825 A.D.) perhaps developed from Hindu term *karaṇī.*

Śrīpati (1039) defines *karaṇī* [*Si.Se.*xiv-7(i)] as follows:

ग्राह्यं न मूलं खलु यस्य राशेस्तस्य प्रदिष्टं करणीति नाम ।

"The number whose square-root cannot be obtained (exactly) is said to form (*karaṇī*) an irrational quantity."

Similar definitions are given by Nārāyaṇa (1356) [NBi. I. R. 25] and others.

Gaṇeśa , in his *Buddhivilāsinī* (p.131), explains the term *karaṇī* as follows :

यस्य मूले गृह्यमाणे सम्यङ्मूलं न लभ्यते तन्नाम करणीति । तदुपचाराद्वर्गराशेरपि मूले गृह्यमाणे सा करणीत्युच्यते । पूर्वेषां पारिभाषिकीयं संज्ञा । तथा चाऽऽहुः – मूलं ग्राह्यं राशेस्तु यस्य करणीति नाम, तस्य स्यादिति ।

"A certain number ,whose square-root is being extracted but cannot be obtained exactly, is called *karaṇī.* In its secondary application, even a square number, when

2. Bag, A. K. 'Mathematichs in ancient and Medieval India' Chuakhambha, Oriental Research Studies, No.16, Varanasi, 1979. p.97.f.n.-1.

its square root is being extracted, is called *karaṇī*. This is a technical term of our predecessors. Thus, they say : 'The name of the number whose square-root is to be taken is *karaṇī*'."

Kṛṣṇa, in his *Bījapallava*, explains the term *karaṇī* (p.50) as:

तत्र यस्य राशेर्मूलेऽपेक्षिते निरग्रं मूलं न संभवति स करणी ।

" Here, the number for which a square-root is needed but cannot be evaluated, is *karaṇī*."

Of course, the number is to be considered a surd when the business is with its square root. A surd number is indicated by putting down the tachygraphic abbreviation *ka* before the number affected. Thus *ka 8* means $\sqrt{8}$ and $ka\ 450\ means\ \sqrt{450}$.

Takao Hayashi in his edition of The *Bakshālī* Manuscript (pp.60-64) traces the term *karaṇī* elaborately.

(v) करणीषड्विधम्

The six operations on the surds[1] :

अथ करणीषड्विधम् –

मूलं ग्राह्यं राशेर्यस्य तु करणीति नाम तस्य स्यात् ॥२५(i)॥

5.1 : Definition of a surd :

" The name of the number whose square-root is to be taken is *karaṇī*." ॥ **25(i)** ॥

The rules framed for addition and subtraction of the surds are based on the following identiy which is explicitly stated below in verse 29 :

$$\sqrt{a} \pm \sqrt{b} = \sqrt{\{(a+b) \pm 2\sqrt{a \cdot b}\}}$$

or

$$\left(\sqrt{a} \pm \sqrt{b}\right)^2 = \{(a+b) \pm 2\sqrt{a \cdot b}\}$$

Let 'a' and 'b' be the two (original) numbers of the surds $\sqrt{a}$, and $\sqrt{b}$. Suppose the sum of the two numbers (of the surds) $(a+b)$ as the *mahati* ('greater'), and twice the square-root of their product $(2\sqrt{ab})$ as the *laghu* ('lesser'). The addition or subtraction of these greater (*mahati*) and lesser (*laghu*) quantities is like that of the original quantities 'a' and 'b' (i.e., integers).

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1 . Datta B. B. and Singh A.N (Revised by Kripashankar Shukla) : 'SURDS IN HINDU MATHAMATICS' : *Indian Journal of History of Science (IJHS)* **28** (3), 1993. 254-264. see pages 254-255.

5.2 : Rules for addition and subtraction of surds :

सङ्गुणनं भजनं वा कुर्याद् वर्गस्य वर्गेण ॥२५॥
लघ्व्या वापि महत्या पृथक् करण्यौ हृते च तत्पदयोः ।
युतिवियुति कृती च तया गुणिते योगान्तरे भवतः ॥२६॥

5.2.1 : "Multiply and divide as if a square number by a square number. Divide the two surds separately by the smaller or greater among them; add or subtract the square-roots of the quotients; then multiply the square of the result by that divisor. The product is the sum or difference." ॥**26** ॥

That is,

$$\sqrt{a} \pm \sqrt{b} = \left[\left\{\sqrt{\left(\frac{a}{b}\right)} \pm \sqrt{\left(\frac{b}{b}\right)}\right\}^2 \cdot b\right]^{1/2}$$

or

$$\sqrt{a} \pm \sqrt{b} = \left[\left\{\sqrt{\left(\frac{b}{b}\right)} \pm \sqrt{\left(\frac{a}{a}\right)}\right\}^2 \cdot a\right]^{1/2}$$

This very rule is stated in its another form in verse 28.

गुणिते वापि करण्यावनल्पया वाऽल्पया च तत्पदयोः ।
युतिवियुतिकृती भक्ते ह्यभीष्टया योगविवरे स्तः ॥२७॥

5.2.2 : "Or multiply the two surds by the smaller or greater one among them; add or subtract the square-roots of the products; then dividing the square of the result by that selected multiplier, the quotient is the sum or difference." ॥**27** ॥

That is,

$$\sqrt{a} \pm \sqrt{b} = \left[\frac{1}{a}\left\{\sqrt{a \cdot a} \pm \sqrt{a \cdot b}\right\}^2\right]^{1/2}$$

$$= \left[\frac{1}{a}\left\{a \pm \sqrt{a \cdot b}\right\}^2\right]^{1/2}$$

or

$$\sqrt{a} \pm \sqrt{b} = \left[\frac{1}{b}\left\{\sqrt{b \cdot a} \pm \sqrt{b \cdot b}\right\}^2\right]^{1/2}$$

$$= \left[\frac{1}{b}\left\{\sqrt{b \cdot a} \pm b\right\}^2\right]^{1/2}$$

अथवा लघ्व्या महतीं भक्त्वैतन्मूलमेकयुक्तोनम् ।
स्वघ्नं लघ्व्या गुणितं युतिवियुती स्तो महत्यैवम् ॥२८॥

5.2.3 : "Or divide the greater surd by the smaller one; add unity to or subtract unity from the square-root of the quotient; then multiply the result by itself and also by the smaller quantity. The result is the sum or difference (required). Or proceed in the same way with the greater surd.[2] ॥ **28** ॥

That is, $$\sqrt{a} \pm \sqrt{b} = \sqrt{\left[\left\{\sqrt{\left(\frac{a}{b}\right)} \pm 1\right\}^2 b\right]}$$

or $$\sqrt{a} \pm \sqrt{b} = \sqrt{\left[\left\{\sqrt{\left(\frac{b}{a}\right)} \pm 1\right\}^2 a\right]}$$

where $a > b$.

[2] . Cf. BBi. vs. ॥ ३४(ii) ॥ 30 ॥

रूपवदपि च करण्योर्घातपदेन द्विसङ्गुणेन युतिः ।
युक्तोना युतिवियुति पृथक्स्थितिः स्यान्न घातपदम् ॥२९॥

5.2.4 : "Or add twice the square-root of the product of the two surds, supposed as if rational, to or subtract that from their sum. The result is the sum or difference. If there be no rational root of the product, then the two surds should be stated severally."[3] ॥ **29** ॥

That is,

$$\sqrt{a} \pm \sqrt{b} = \sqrt{\{(a+b) \pm 2\sqrt{ab}\}}.$$

करणीनां तु बहूनां योगे केनापि राशिना छित्वा ।
तन्मूलयुतिः स्वघ्ना छेदगुणा स्याद्युतिस्तासाम् ॥३०॥

5.2.5 : "To add up several surds, divide them by an optional number and then take the sum of the square-root of the quotients. This sum multiplied by itself and also by that divisor will give the sum of them."[4] ॥ **30** ॥

That is,

$$\sqrt{a} \pm \sqrt{b} = \left[c\left\{\sqrt{\left(\frac{a}{c}\right)} + \sqrt{\left(\frac{b}{c}\right)}\right\}^2\right]^{1/2}$$

The optional number c is so chosen that $\left(\frac{a}{c}, \frac{b}{c}\right)$ become perfect squares.

[3] . Cf. BBi. vs. ॥ ३४(i) ॥ 29 ॥
[4] . Cf. Br. Sp. Si. xviii. ; GSS. vii. $88\frac{1}{2}$.

उदाहरणम् –

षट्सिद्धसङ्ख्ययोर्योगविशेषौ वद मे द्रुतम् ।
करण्योर्द्वित्रिमित्योश्च योगशेषे तयोर्वद ॥१५॥

Ex. 15 : "Tell me quickly the sum and difference of the surds : $\sqrt{6}$ and $\sqrt{24}$, and also that of $\sqrt{2}$ and $\sqrt{3}$." ॥ **15** ॥

Statement : Find : (i) $\sqrt{6} + \sqrt{24}$ or $\sqrt{24} - \sqrt{6}$
(ii) $\sqrt{2} + \sqrt{3}$ or $\sqrt{3} - \sqrt{2}$.

Solution : Ex. (i) $\sqrt{6} + \sqrt{24}$ or $\sqrt{24} - \sqrt{6}$
I Method : By the rule stated in verse 26,
Let $a = 24$ and $b = 6$, then

$\sqrt{a} \pm \sqrt{b} = \sqrt{\left\{\sqrt{\left(\frac{a}{b}\right)} \pm \sqrt{\left(\frac{b}{b}\right)}\right\}^2 \cdot b}$	$\sqrt{a} \pm \sqrt{b} = \sqrt{\left\{\sqrt{\left(\frac{b}{b}\right)} \pm \sqrt{\left(\frac{a}{a}\right)}\right\}^2 \cdot a}$
$\frac{a}{b} = \frac{24}{6} = 4, \quad \frac{b}{b} = \frac{6}{6} = 1$ $\sqrt{\frac{a}{b}} = \sqrt{4} = 2, \quad \sqrt{\left(\frac{b}{b}\right)} = \sqrt{1} = 1$ $\left\{\sqrt{\left(\frac{a}{b}\right)} \pm \sqrt{\left(\frac{b}{b}\right)}\right\}^2 = (2 \pm 1)^2$ $\therefore \left(\sqrt{a} + \sqrt{b}\right)^2 = \{6 \times (2+1)^2\}$ $= 54$; and $\left(\sqrt{a} - \sqrt{b}\right)^2 = \{6 \times (1)^2\} = 6$ $\therefore \sqrt{24} \pm \sqrt{6} = (54, \quad 6)$	$\frac{a}{a} = \frac{24}{24} = 1, \quad \frac{b}{a} = \frac{6}{24} = \frac{1}{4}$ $\sqrt{\frac{a}{a}} = \sqrt{1} = 1, \quad \sqrt{\left(\frac{b}{a}\right)} = \sqrt{\frac{1}{4}} = \frac{1}{2}$ $\left\{\sqrt{\left(\frac{a}{a}\right)} \pm \sqrt{\left(\frac{b}{a}\right)}\right\}^2 = \left(1 \pm \frac{1}{2}\right)^2$ $\therefore \left(\sqrt{a} + \sqrt{b}\right)^2 = \left\{24 \times \left(\frac{3}{2}\right)^2\right\}$ $= 54$; and $\left(\sqrt{a} - \sqrt{b}\right)^2 = \left\{24 \times \left(\frac{1}{2}\right)^2\right\} = 6$ $\sqrt{24} \pm \sqrt{6} = (54, \quad 6)$

II Method : By the rule stated in verse 27,
Let $a = 24$ and $b = 6$, then

$\sqrt{a}\pm\sqrt{b}=\sqrt{\left[\frac{1}{a}\left\{a\pm\sqrt{a\cdot b}\right\}^2\right]}$	$\sqrt{a}\pm\sqrt{b}=\sqrt{\left[\frac{1}{b}\left\{\sqrt{b\cdot a}\pm b\right\}^2\right]}$
$a\cdot b = 24\times 6 = 144;$ $\sqrt{a\cdot b}=\sqrt{144}=12;$ $\left\{a\pm\sqrt{a\cdot b}\right\}^2=(24\pm 12)^2$ $=(36^2, 12^2)$ $\frac{1}{a}\left\{a\pm\sqrt{a\cdot b}\right\}^2=\frac{1296}{24},\frac{144}{24}$ $=(54, \quad 6)$ $\therefore \sqrt{24}\pm\sqrt{6}=(54, \quad 6)$	$b\cdot a = 24\times 6 = 144;$ $\sqrt{b\cdot a}=\sqrt{144}=12;$ $\left\{\sqrt{b\cdot a}\pm b\right\}^2=(12\pm 6)^2$ $=(18^2, \quad 6^2)$ $\frac{1}{b}\left\{\sqrt{b\cdot a}\pm b\right\}^2=\frac{324}{6},\frac{36}{6}$ $=(54, \quad 6)$ $\therefore \sqrt{24}\pm\sqrt{6}=(54, \quad 6)$

III Method : By the rule stated in verse 28,
Let $a = 24$ and $b = 6$, then

$\sqrt{a}\pm\sqrt{b}=\sqrt{\left[\left\{\sqrt{\left(\frac{a}{b}\right)}\pm 1\right\}^2 b\right]}$	$\sqrt{a}\pm\sqrt{b}=\sqrt{\left[\left\{\sqrt{\left(\frac{b}{a}\right)}\pm 1\right\}^2 a\right]}$
$\frac{a}{b}=\frac{24}{6}=4,\ \therefore \sqrt{\frac{a}{b}}=\sqrt{4}=2,$ $\left\{\sqrt{\left(\frac{a}{b}\right)}\pm 1\right\}^2 b = 6(2\pm 1)^2$ $\therefore(\sqrt{a}+\sqrt{b})^2=\{6\times(2+1)^2\}$ $=54;$ and $(\sqrt{a}-\sqrt{b})^2=\{6\times(1)^2\}=6$ $\therefore \sqrt{24}\pm\sqrt{6}=(54, \quad 6)$	$\frac{b}{a}=\frac{6}{24}=\frac{1}{4}\ ;\ \sqrt{\left(\frac{b}{a}\right)}=\sqrt{\frac{1}{4}}=\frac{1}{2}$ $\left\{\sqrt{\left(\frac{b}{a}\right)}\pm 1\right\}^2 a = 24\left(1\pm\frac{1}{2}\right)^2$ $\therefore(\sqrt{a}+\sqrt{b})^2=\left\{24\times\left(\frac{3}{2}\right)^2\right\}$ $=54;$ and $(\sqrt{a}-\sqrt{b})^2=\left\{24\times\left(\frac{1}{2}\right)^2\right\}=6$ $\sqrt{24}\pm\sqrt{6}=(54, \quad 6)$

IV Method : By the rule stated in verse 29,
Let $a = 24$ and $b = 6$, then

$$\left(\sqrt{a}\pm\sqrt{b}\right)^2 = \left\{(a+b)\pm 2\sqrt{a\cdot b}\right\}$$
$$= \left\{(24+6)\pm 2\sqrt{24\times 6}\right\}$$
$$= 30\pm 24$$
$$= (54,\ 6).$$

or $\quad \sqrt{a}\pm\sqrt{b}=\sqrt{\left\{(a+b)\pm 2\sqrt{a\cdot b}\right\}}$

$\therefore\ \sqrt{24}\pm\sqrt{6}=\left(\sqrt{54},\ \sqrt{6}\right).$

Ex. (ii) $\mathbf{\sqrt{2}+\sqrt{3}}$ **or** $\mathbf{\sqrt{3}-\sqrt{2}.}$

Let $a = 3$ and $b = 2$; here, there is no rational root of the product $\sqrt{a\cdot b}=\sqrt{3\cdot 2}$, therefore in accoradance with the rule stated in verse 29, the two surds should be stated severally.

$\therefore\ \mathbf{\sqrt{2}+\sqrt{3}=\sqrt{2}+\sqrt{3}}$ **and** $\mathbf{\sqrt{3}-\sqrt{2}=\sqrt{3}-\sqrt{2}\ .}$

इति करणीसङ्कलनव्यवकलने ।

Thus ends the operations of addition and subtraction of surds.

करणीगुणनादौ सूत्रम् –

करणीनां (च) बहूनां यासां संयोगसंभवोऽप्यस्ति ।
तासां योगं कृत्वा कार्यं गुणनादि वा कर्म ॥३१॥

5.3 : Rule for multiplication of surds :

"When there exist many component surd terms in a surd expression, and addition (or subtraction) of them is possible, (for abridgement,) multiplication or division of the surd expression should proceed with after addition of two or more terms of the multiplier and multiplicand "॥**31**॥

क्षयरूपकृतिः क्षयगा भवेद्यदा सा प्रयाति करणीत्वम् ।
क्षयगतकरणीमूलं रूपत्वं क्षयगतं भवति ॥३२॥

"The squaring of a negative rational quantity (*rūpa*) should be negative or of the divisor and dividend.if it is achieved for the purpose of its being a surd. Like wise the square-root of a surd having the nature of a negative, is negative for the sake of the creation of *rūpa* (rational number)."[5] ॥ **32** ॥

This is a seeming exception to the maxim, 'that a negative quantity has no square-root'[vide,vs.No.10]. But the sign belongs to the surd root not to the entire irrational quantity. When therefore a negative rational quantity is squared to bring it to the same form with a surd, with which it is to be combined, it retains the negative sign

[5] . Cf. BBi. ॥३७॥ 33 ॥

appertaining to the root : and in like manner, when a root is extracted out of a negative rational part of a compound surd, the root has the negative sign.

उदाहरणम् –

षड्रूपाढ्या गणक! करणीपञ्चसङ्ख्या च गुण्यो
द्वेऽश्टौ पञ्च–प्रमितकरणीखण्डसङ्ख्या गुणश्च ।
षड्रूपोन शरनखमिते वा गुणे किं फलं स्यात्
तद्गुण्याप्तां वद गुणमितिं प्रौढता चेत्तवास्ति ॥१६॥

Ex. 16 : "Oh mathematician ! The multiplicand is $(6+\sqrt{5})$ and the multiplier is $(\sqrt{8}+\sqrt{5}+\sqrt{2})$; or with the same multiplicand, and $(-6+\sqrt{20}+\sqrt{5})$ as the multiplier, tell me, what will be the products if you are proficient in the operation of multiplication" ॥ **16** ॥

Solution :

Statement :Ex. (i). multiplicand : $(6+\sqrt{5})$; or $(\sqrt{36}+\sqrt{5})$
multiplier $(\sqrt{8}+\sqrt{5}+\sqrt{2})$.

Method –I :

In accordance with the rule stated in verse 21-22, here, the method of multiplication by component parts is to be adopted :

$$(\sqrt{36}+\sqrt{5})\times\sqrt{8}=\sqrt{288}+\sqrt{40}$$
$$(\sqrt{36}+\sqrt{5})\times\sqrt{5}=\sqrt{180}+\sqrt{25}$$
$$\underline{(\sqrt{36}+\sqrt{5})\times\sqrt{2}=\sqrt{72}+\sqrt{10}\,.}$$

$$\begin{aligned}\therefore \text{Product} &= (\sqrt{288}+\sqrt{72})+(\sqrt{40}+\sqrt{10})+\sqrt{180}+\sqrt{25}\\ &= (12\sqrt{2}+6\sqrt{2})+(2\sqrt{10}+\sqrt{10})+\sqrt{180}+5\\ &= 18\sqrt{2}+3\sqrt{10}+\sqrt{180}+5\\ &= 5+\sqrt{648}+\sqrt{180}+\sqrt{90}\,.\end{aligned}$$

As stated in verse 31, and later in the commentary under verse 38, :

करणिखण्डेषु ययोर्ययोर्योगसम्भवोऽप्यस्ति तयो (र्तयो) र्योगं कृत्वा गुणन–भजन–वर्गवर्गमूलानि कार्याणि ।

"Multiplication, division, squaring or extraction of square-root of the surd expression, should proceed with after addition of two or more terms of the multiplier and multiplicand or of the divisor and dividend where ever it is possible."

Method –II : In the multiplier of the present example :

$$\sqrt{8} + \sqrt{2} = 2\sqrt{2} + \sqrt{2}$$
$$= 3\sqrt{2} = \sqrt{18}.$$

∴ abridged multiplier : $\left(\sqrt{8} + \sqrt{5} + \sqrt{2}\right) = \sqrt{18} + \sqrt{5}$

$\left(\sqrt{36} + \sqrt{5}\right) \times \sqrt{18} = \sqrt{648} + \sqrt{90}$

$\underline{\left(\sqrt{36} + \sqrt{5}\right) \times \sqrt{5} = \sqrt{180} + \sqrt{25}}$

Product : $5 + \sqrt{648} + \sqrt{180} + \sqrt{90}$.

Ex. (ii). multiplicand : $\left(6 + \sqrt{5}\right)$; or $\left(\sqrt{36} + \sqrt{5}\right)$

multiplier: $\left(-\sqrt{36} + \sqrt{20} + \sqrt{5}\right)$.

$\left(\sqrt{36} + \sqrt{5}\right) \times \sqrt{-36} = -\sqrt{1296} - \sqrt{180}$

$\left(\sqrt{36} + \sqrt{5}\right) \times \sqrt{20} = \sqrt{720} + \sqrt{100}$

$\underline{\left(\sqrt{36} + \sqrt{5}\right) \times \sqrt{5} = \sqrt{180} + \sqrt{25}}$

Product : $-\sqrt{1296} + \sqrt{100} + \sqrt{25} + \left(\sqrt{180} - \sqrt{180}\right) + \sqrt{720}$

$= -36 + 10 + 5 + \sqrt{720}$

$= -21 + \sqrt{720}.$

Since $\sqrt{20} + \sqrt{5} = 2\sqrt{5} + \sqrt{5}$

$= 3\sqrt{5} = \sqrt{45}$,

the abridged multiplier = $-\sqrt{36} + \sqrt{45}$

$\left(\sqrt{36} + \sqrt{5}\right) \times -\sqrt{36} = -\sqrt{1296} - \sqrt{180}$

$\underline{\left(\sqrt{36} + \sqrt{5}\right) \times \sqrt{45} = \sqrt{36 \times 45} + \sqrt{5 \times 45}}$

Product : = $-\sqrt{1296} - \sqrt{180} + \sqrt{36 \times 45} + \sqrt{225}$

$= -36 - 2\sqrt{45} + 6\sqrt{45} + 15$

$= -21 + 4\sqrt{45}$

$= -21 + \sqrt{720}.$

अपि च –

रूपद्वयाढ्यकरणीद्वितयेन निघ्ना
दन्तस्मृतीभयुगलप्रमिताः करण्यः ।
किं स्यात्फलं कथय तत् त्वरितं करण्या
निघ्ना भुजङ्गयमलोन्मितयाऽथवा ताः ॥१७॥

Ex.17 : "$(\sqrt{32} + \sqrt{18} + \sqrt{8} + \sqrt{2})$ is multiplied by $(2 + \sqrt{2})$, or (when the same is multiplied) by $(\sqrt{8} + \sqrt{2})$ what will be the product of those? Tell (me) quickly." ॥ **17** ॥

Statement (*Nyāsa*): Multiplicand : $(\sqrt{32} + \sqrt{18} + \sqrt{8} + \sqrt{2})$;

(i) Multiplier : $(2 + \sqrt{2})$ or $\left(\sqrt{4}, \sqrt{2}\right)$.

(ii) Multiplier : $\left(\sqrt{8}, \sqrt{2}\right)$.

Solution : Case-1

$(\sqrt{32} + \sqrt{18} + \sqrt{8} + \sqrt{2}) \times \sqrt{4} = \sqrt{128} + \sqrt{72} + \sqrt{32} + \sqrt{8}$

$\underline{\sqrt{32} + \sqrt{18} + \sqrt{8} + \sqrt{2}) \times \sqrt{2} = \sqrt{64} + \sqrt{36} + \sqrt{16} + \sqrt{4}}$

Product: $\left(\sqrt{128} + \sqrt{72} + \sqrt{32} + \sqrt{8}\right) + \left(\sqrt{64} + \sqrt{36} + \sqrt{16} + \sqrt{4}\right)$.

Here, $\sqrt{64}+\sqrt{36}+\sqrt{16}+\sqrt{4} = 8+6+4+2 = 20$;
and the remaining surds are $(\sqrt{128}+\sqrt{72}+\sqrt{32}+\sqrt{8})$.

Now, in accordance with the rule stated in verse 30, dividing by the optional number 2 and taking the roots:

$$(\sqrt{128}+\sqrt{72}+\sqrt{32}+\sqrt{8}) = \sqrt{2}(\sqrt{64}+\sqrt{36}+\sqrt{16}+\sqrt{4})$$
$$= \sqrt{2}\,(20) = \sqrt{800}.$$

$\therefore$ The product = $20+\sqrt{800}$.

Or

Multiplicand : $(\sqrt{32}+\sqrt{18}+\sqrt{8}+\sqrt{2})$;

Multiplier : $(2+\sqrt{2})$ or $(\sqrt{4},\sqrt{2})$.

Abridging the Multiplicand : $\sqrt{32}+\sqrt{18}+\sqrt{8}+\sqrt{2}$;

$$\sqrt{32}+\sqrt{18}+\sqrt{8}+\sqrt{2} = 4\sqrt{2}+3\sqrt{2}+2\sqrt{2}+\sqrt{2}$$
$$= 10\sqrt{2}$$
$$= \sqrt{200}.$$

$\therefore$ Multiplicand : $\sqrt{200}$; Multiplier : $(\sqrt{4},\sqrt{2})$.

$$\therefore Product = (\sqrt{200}\times\sqrt{4}) + (\sqrt{200}\times\sqrt{2})$$
$$= \sqrt{800}+\sqrt{400}$$
$$= 20+\sqrt{800}.$$

: Case-2 :

Multiplicand : $\sqrt{200}$; Multiplier : $(\sqrt{8},\sqrt{2})$.

$$\therefore Product = (\sqrt{200}\times\sqrt{8}) + (\sqrt{200}\times\sqrt{2})$$
$$= \sqrt{1600}+\sqrt{400}$$
$$= 40+20 = 60.$$

इति करणीगुणनम् ।

Thus ends the multiplication of surds

करणीभागहारे सूत्रम् –

छेदकरण्यो निखिलाः कत्यपि वा वर्गसिद्धये हन्यात् ।
तासां मूलसमासो रूपसमानो यथा भवति ॥३३॥
लब्धकरण्यो गुणकस्तद्गुणहारं त्यजेद् भाज्यात् ।
शुद्धिर्न भवेद्यदि वा तद्गुणकच्छेदकरणीनाम् ॥३४॥
योगमपि भाज्यराशेर्जह्यात्सदृशखण्डे च ।
रूपाभावे हारो येन निघ्नः शुध्यति तत्फलं स्यात् ॥३५॥

5. 4 : Rule for division of surds :

"Multiply the divisor surd so as to make all or some of its terms square such that the sum of their square-roots will be equal to the rational term (in the dividend). Thus will be determined the multiplier surd. Subtract from the dividend the divisor multiplied by that. If there be left a remainder, the sum of the terms of the divisor multiplied by that multiplier should be subtracted from the terms of the dividend. In case of absence of a rational term (in the dividend), that by which the divisor is multiplied and then subtracted for the dividend so as to leave no remainder, will be the quotient." [6] ॥ **33-35** ॥

Illistrative examples :

(i). Divide: $(5+\sqrt{648}+\sqrt{180}+\sqrt{90})$ *by* $(\sqrt{36}+\sqrt{5})$.

(ii). Divide : $-21+\sqrt{720}$ by $(\sqrt{36}+\sqrt{5})$.

(iii) : Divide : $20+\sqrt{800}$ by $\sqrt{4}+\sqrt{2}$

(iv) : Divide 60 (or $\sqrt{3600}$) by $\sqrt{18}$.

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[6] . IJHS, **28**(3), 1993, p. 256, f.n.19

Example (i):

Divide $\left(5+\sqrt{90}+\sqrt{180}+\sqrt{648}\right)\ by\ \left(\sqrt{5}+\sqrt{36}\right)$

$\sqrt{5}+\sqrt{36}\ \big)\ 5+\sqrt{90}+\sqrt{180}+\sqrt{648}\ \big(\ \sqrt{5}+\sqrt{18}$

$\underline{5 \qquad +\sqrt{180}}$

$\sqrt{90}+ \qquad +\sqrt{648}$

$\underline{\sqrt{90}+ \qquad +\sqrt{648}}$

Example (ii) : Divide $-21+\sqrt{720}$ by $\left(\sqrt{5}+\sqrt{36}\right)$

As stated in verse 33, : "Multiply the divisor surd so as to make all or some of its terms square such that the sum of their square-roots will be equal to the rational term (in the dividend).

$\sqrt{5}+\sqrt{36}\ \big) \qquad -21+\sqrt{720} \quad \big(\quad -\sqrt{36}+\sqrt{45}$

$\underline{-\sqrt{36\times 36}-\sqrt{180}}$

$(15)+\left(\sqrt{180}+\sqrt{720}\right)=18\sqrt{5}$

$\underline{\left(\sqrt{5\times 45}\right)=(15)+\left(\sqrt{36\times 45}\right)= \qquad 18\sqrt{5}}$

Or

$\sqrt{5}+\sqrt{36}\ \big)\ -21+\sqrt{720} \qquad \big(\sqrt{20}-\sqrt{36}+\sqrt{5}$

$\underline{\qquad \sqrt{720}\ +\sqrt{100}}$

$-21 \qquad -\sqrt{100}$

$\underline{-36\ \ -\sqrt{180} \qquad}$

$15+\sqrt{180}-\sqrt{100}$

or $\quad 5+\sqrt{180}$

$\underline{5+\sqrt{180}.}$

Example (iii) : Divide $20+\sqrt{800}$ by $\sqrt{4}+\sqrt{2}$

$\sqrt{2}+\sqrt{4}\ \big)\ 20+\sqrt{800}\ \big(\ \sqrt{200}$

$\underline{\sqrt{400}+\sqrt{800}}$

$\sqrt{200}=10\sqrt{2}$

$=(4+3+2+1)\sqrt{2}$

$=\sqrt{32}+(18)+\sqrt{8}+\sqrt{2}.$

Example (iv) : Divide 60 (or $\sqrt{3600}$) by $\sqrt{18}$.

$\sqrt{18}\ \big)\ \sqrt{3600}\ \big(\ \sqrt{200}$

$\underline{\sqrt{3600}}$

$\sqrt{200}=\sqrt{32}+(18)+\sqrt{8}+\sqrt{2}.$

करणीविश्लेषणे सूत्रम् –

योगजकरणीं कृत्वा कयाऽपि शुद्धिर्यथा भवेद् विभजेत् ।
तत्खण्डानि स्वगुणानि लब्ध्यां हतानि च करण्यः ॥३६॥

5.4.1 : Rule of separation of component surds in a compound-surd

"Divide the compound-surd by the square of some number so as to leave no remainder, Parts of it multiplied by themselves and also by the quotient will be the (component) terms of the surd."[7] ॥ **36** ॥

That is to say, if $N = m^2k$ and $m = a + b + c + d$.

Then, $\sqrt{N} = \sqrt{m^2k} = m\sqrt{k} = (a + b + c + d)\sqrt{k}.$

$= \sqrt{a.a.k} + \sqrt{b.b.k} + \sqrt{c.c.k} + \sqrt{d.d.k}$.

$= \sqrt{a^2k} + \sqrt{b^2k} + \sqrt{c^2k} + \sqrt{d^2k}$.

Illustrative Examples:

(i). $\sqrt{18} + \sqrt{5} = \sqrt{5} + \sqrt{9 \times 2}$

$= \sqrt{5} + 3\sqrt{2} = \sqrt{5} + 2\sqrt{2} + 1\sqrt{2}$

$= \sqrt{5} + \sqrt{8} + \sqrt{2}.$

(ii). $-\sqrt{36} + \sqrt{45} = -\sqrt{36} + \sqrt{9 \times 5}$

$= -\sqrt{36} + 3\sqrt{5} = -\sqrt{36} + 2\sqrt{5} + 1\sqrt{5}$

$= -\sqrt{36} + \sqrt{20} + \sqrt{5}.$

(iii) $\sqrt{200} = 10\sqrt{2}$

$= 4\sqrt{2} + 3\sqrt{2} + 2\sqrt{2} + 1\sqrt{2}$

$= \sqrt{32} + \sqrt{18} + \sqrt{8} + \sqrt{2}.$

[7] . Cf. BBi. R. ॥३९/36॥; IJHS, **28**(3), 1993, p. 258, f.n.22-23.

अपि च –

बाणाग्नयः खदहनाः शशिलोचनानि
वस्विन्दवोऽब्धिशशिनो यमलैन्दवश्च ।
खण्डानि तानि करणीशरताडितानि
द्वित्रीषुखण्डविहृतानि सखे फलं किम् ॥१८॥

EX. 18 : "$\{(\sqrt{35} + \sqrt{30} + \sqrt{21} + \sqrt{18} + \sqrt{14} + \sqrt{12}) \times \sqrt{5}\}$ is divided by $(\sqrt{2} + \sqrt{3} + \sqrt{5})$, Oh friend ! (Tell me) what is the quotient ?" ॥ **18** ॥

Solution:

Statement (*Nyāsa*) :

Dividend : $\{(\sqrt{35} + \sqrt{30} + \sqrt{21} + \sqrt{18} + \sqrt{14} + \sqrt{12}) \times \sqrt{5}\}$;

or $(\sqrt{175} + \sqrt{150} + \sqrt{105} + \sqrt{90} + \sqrt{70} + \sqrt{60})$

Divisor : $(\sqrt{5} + \sqrt{3} + \sqrt{2})$.

Method - I : In the given problem, there is no rational term. Hence, as stated in the verse 35, :

"In case of absence of a rational term (in the dividend), that by which the divisor is multiplied and then subtracted for the dividend so as to leave no remainder, will be the quotient."

$$\sqrt{5} + \sqrt{3} + \sqrt{2}\,)\ \sqrt{175} + \sqrt{150} + \sqrt{105} + \sqrt{90} + \sqrt{70} + \sqrt{60}\ (\sqrt{35} + \sqrt{30}$$

$$\underline{\sqrt{175} + \quad \sqrt{105} + \quad \sqrt{70}}$$

$$\sqrt{150} + \quad \sqrt{90} + \quad \sqrt{60}$$

$$\underline{\sqrt{150} + \quad \sqrt{90} + \quad \sqrt{60}}$$

$\therefore$ The quotient obtained is : $\sqrt{35} + \sqrt{30}$.

अथवाऽन्यथोच्यते –

छेदेऽभीष्टकरण्या ऋणधनताव्यत्ययोऽसकृत्कार्यः ।
भाज्यहरौ सङ्गुणयेद्यावच्छेदे करण्यैका ॥३७॥
विभजेत्तया करण्या भाज्योद्भूताः करण्यश्च ।
लब्धा योगजकरणी चेत् स्याद्विश्लेषणं प्राग्वत् ॥३८॥

5.4.2: Another rule for division of surds (By Rationalisation of the denominator)[8] :

"Reversing the sign, negative or positive, of one of the surds occurring in the denominator, multiply by it, both the numerator and the denominator separately, until there remains only one surd in the denominator.

The surds which constituted the dividend, are to be divided by that single remaining surd: and if the surds obtained as quotient be such as arise from addition, they must be separated by the preceding rule for separation of them, in such form as the questioner may desire."॥ **37-38** ॥

Example (i): Divide: $\left(5+\sqrt{648}+\sqrt{180}+\sqrt{90}\right)$ $by\left(\sqrt{36}+\sqrt{5}\right)$.

$$= \frac{\left(\sqrt{25}+\sqrt{648}+\sqrt{180}+\sqrt{90}\right)}{\left(\sqrt{36}+\sqrt{5}\right)} \times \frac{\left(\sqrt{36}-\sqrt{5}\right).}{\left(\sqrt{36}-\sqrt{5}\right).}$$

$$= \frac{\sqrt{900}+\sqrt{648\times 36}+\sqrt{180\times 36}+\sqrt{90\times 36}-\sqrt{125}-\sqrt{648\times 5}-\sqrt{900}-\sqrt{450}}{\sqrt{961}}$$

$$= \left\{\sqrt{36\times 36\times 9\times 2}\right\}+\sqrt{36\times 36\times 5}-\sqrt{5\times 5\times 5}-\sqrt{15\times 15\times 2}\} \div \sqrt{961}$$

$$= (108\sqrt{2}-15\sqrt{2}+36\sqrt{5}-5\sqrt{5}) \div \sqrt{961})$$

[8] . Cf. Br,Sp.Si. XViii, 39. IJHS, **28**(3), 1993, p. 258, f.n.16-18.

$$= (93\sqrt{2}+31\sqrt{5}) \div \sqrt{961}$$

$$= \left(\sqrt{17298}+\sqrt{4805}\right) \div \sqrt{961}$$

$$= \sqrt{18}+\sqrt{5}$$

$$= \sqrt{8}+\sqrt{2}+\sqrt{5}\,.$$

(ii). Divide : $\mathbf{-21+\sqrt{720}}$ **by** $\left(\sqrt{\mathbf{36}}+\sqrt{\mathbf{5}}\right)$**.**

$$= \frac{\left(-\sqrt{441}+\sqrt{720}\right)}{\left(\sqrt{36}+\sqrt{5}\right)} \times \frac{\left(\sqrt{36}-\sqrt{5}\right)}{\left(\sqrt{36}-\sqrt{5}\right)}$$

$$= \frac{-\sqrt{441\times 36}+\sqrt{720\times 36}+\sqrt{441\times 5}-\sqrt{720\times 5}}{\sqrt{961}}$$

$$= \left\{(-21\times 6-60)+\left(24\sqrt{45}+7\sqrt{45}\right)\right\} \div \left(\sqrt{961}\right)$$

$$= \left(-186+31\sqrt{45}\right) \div \left(\sqrt{961}\right)$$

$$= -6+\sqrt{45} = -6+3\sqrt{5} = -6+2\sqrt{5}+1\sqrt{5}$$

$$= -6+\sqrt{20}+\sqrt{5}\,.$$

(iii) : Divide : $\mathbf{20+\sqrt{800}}$ **by** $\sqrt{\mathbf{4}}+\sqrt{\mathbf{2}}$

$$= \frac{\sqrt{400}+\sqrt{800}}{\sqrt{4}+\sqrt{2}} \times \frac{\left(-\sqrt{4}+\sqrt{2}\right)}{\left(-\sqrt{4}+\sqrt{2}\right)}$$

$$= \frac{-\sqrt{1600}-\sqrt{3200}+\sqrt{800}+\sqrt{1600}}{-\sqrt{4}}$$

$$= \frac{-40\sqrt{2}+20\sqrt{2}}{-\sqrt{4}} = \frac{-20\sqrt{2}}{-\sqrt{4}} = \frac{\sqrt{800}}{\sqrt{4}} = \sqrt{200}\,.$$

(iv) : Divide 60 (or $\sqrt{3600}$) by $\sqrt{18}$.

$$= \frac{\sqrt{3600}}{\sqrt{18}} = \sqrt{200}$$

Ex. 18 : Statement (*Nyāsa*) :

Dividend : $\{(\sqrt{35}+\sqrt{30}+\sqrt{21}+\sqrt{18}+\sqrt{14}+\sqrt{12})\times\sqrt{5}\}$;

or $(\sqrt{175}+\sqrt{150}+\sqrt{105}+\sqrt{90}+\sqrt{70}+\sqrt{60})$

Divisor : $(\sqrt{5}+\sqrt{3}+\sqrt{2})$.

Method- II. Division by rationalising the denominator[9] :

$$(\sqrt{175}+\sqrt{150}+\sqrt{105}+\sqrt{90}+\sqrt{70}+\sqrt{60})\div(\sqrt{5}+\sqrt{3}+\sqrt{2}).$$

$$=\frac{(\sqrt{175}+\sqrt{150}+\sqrt{105}+\sqrt{90}+\sqrt{70}+\sqrt{60})(\sqrt{5}+\sqrt{3}-\sqrt{2})}{(\sqrt{5}+\sqrt{3}+\sqrt{2})(\sqrt{5}+\sqrt{3}-\sqrt{2})}.$$

$$=\frac{\begin{array}{l}(\sqrt{875}+\sqrt{750}+\sqrt{525}+\sqrt{450}+\sqrt{350}+\sqrt{300})+\\(\sqrt{525}+\sqrt{450}+\sqrt{315}+\sqrt{270}+\sqrt{210}+\sqrt{180})+\\-\sqrt{350}-\sqrt{300}-\sqrt{210}-\sqrt{180}-\sqrt{140}-\sqrt{120}\end{array}}{5+3+2\sqrt{15}-2}.$$

$$=\frac{\begin{array}{c}\sqrt{5^2\cdot35}+\sqrt{5^2\cdot30}+\sqrt{525}+\sqrt{525}+\sqrt{450}+\sqrt{450}\\+\sqrt{3^2\cdot35}+\sqrt{3^2\cdot30}-\sqrt{2^2\cdot35}-\sqrt{2^2\cdot30}\end{array}}{\sqrt{60}+\sqrt{6^2}}.$$

$$=\frac{\sqrt{(5+3-2)^2\cdot35}+\sqrt{(5+3-2)^2\cdot30}+\sqrt{2^2\cdot525}+\sqrt{2^2\cdot450}}{\sqrt{60}+\sqrt{36}}.$$

$$=\frac{\sqrt{1260}+\sqrt{1080}+\sqrt{2100}+\sqrt{1800}}{(\sqrt{60}+\sqrt{36})}\times\frac{(\sqrt{60}-\sqrt{36})}{(\sqrt{60}-\sqrt{36})}.$$

$$=\frac{\begin{array}{c}\sqrt{75600}+\sqrt{64800}+\sqrt{126000}+\sqrt{10800}\\-\sqrt{45360}-\sqrt{38880}-\sqrt{75600}-\sqrt{64800}\end{array}}{60-36}.$$

$$=\frac{\sqrt{126000}+\sqrt{10800}-\sqrt{45360}-\sqrt{38880}}{24}.$$

[9] . IJHS, **28**(3), 1993, p. 257 , Second method, f.n.21.

$$=\frac{60\sqrt{35}+60\sqrt{30}-36\sqrt{35}-36\sqrt{30}}{24}=\frac{24\sqrt{35}+24\sqrt{30}}{24}.$$

$$=\sqrt{35}+\sqrt{30}.$$

इति करणीभागहारः ।

Thus ends the division of surds.

अथ करणीवर्गे सूत्रम् –

धनगतकरणीनां वा क्षयगानां तत्समानरूपाणि ।
स्युर्धनगतानि वर्गे शेषाः स्वमृणोपलक्ष्यास्ताः ॥३९॥

अन्तरकरणीवर्गे सम्प्राप्तेऽपि च तयोर्धनक्षययोः ।
व्यस्तं स्वर्णं सुधिया कार्यं वर्गस्तयोस्तुल्यः ॥४०॥

5.5 : Rule for squaring a surd :

"In the square (of a surd expression), the square terms of either a positive component surd, or of a negative component surd, is to be treated similarly as positive only; and the rest of the terms should be indicated with their proper sign.

In the square of a surd expression, where the difference of surds is taken : the square of the given expression, and also that in which the positive is made negative and the negative positive (i.e., on reversing the sign of the components surds), and the operation is carried out, the learned should know, the results obtained are equal (or one and the same in each case)." ॥ **39-40** ॥

उदाहरणम् –

> वर्गं करण्योर्द्विकराममित्यो–
> स्त्रिषड् द्विकानां च पृथग्वदाशु ॥
> सप्तद्विपञ्चत्रिकसम्मिताना–
> मङ्गेषुरामाक्षिमहीमितानाम् ।
> दन्तस्मृतीभद्विकसम्मितानां
> चेद् वेत्सि विद्वन करणीविधानम् ॥१९॥

Ex. 19 : **"(**Find) the square of : $(\sqrt{3}+\sqrt{2})$; $(\sqrt{6}+\sqrt{3}+\sqrt{2})$; $(\sqrt{7}+\sqrt{5}+\sqrt{3}+\sqrt{2})$; $(\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}+1)$; and $(\sqrt{32}+\sqrt{18}+\sqrt{8}+\sqrt{2})$, if you know the operation of squaring of surds." II **19** II

Solution :

(i). $(\sqrt{3}+\sqrt{2})^2 = 3+2+2(\sqrt{3}\times\sqrt{2})$
$= 5+\sqrt{24}.$

(ii). $(\sqrt{6}+\sqrt{3}+\sqrt{2})^2 = 6+3+2+2\sqrt{6}\sqrt{3}+2\sqrt{3}\sqrt{2}+2\sqrt{2}\sqrt{6}.$
$= 11+\sqrt{72}+\sqrt{24}+\sqrt{48}.$

(iii). $(\sqrt{7}+\sqrt{5}+\sqrt{3}+\sqrt{2})^2 = 7+5+3+2+2\sqrt{7}\sqrt{5}+$
$2\sqrt{5}\sqrt{3}+2\sqrt{3}\sqrt{2}+2\sqrt{2}\sqrt{7}+2\sqrt{7}\sqrt{3}+2\sqrt{5}\sqrt{2}.$
$= 17+\sqrt{140}+\sqrt{60}+\sqrt{24}+\sqrt{56}+\sqrt{84}+\sqrt{40}.$
$= 17+\sqrt{140}+\sqrt{84}+\sqrt{60}+\sqrt{56}+\sqrt{40}+\sqrt{24}.$

(iv). $(\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}+1)^2 = 6+5+3+2+1+2\sqrt{6}\sqrt{5}+$
$2\sqrt{6}\sqrt{3}+2\sqrt{6}\sqrt{2}+2\sqrt{6}\sqrt{1}+2\sqrt{5}\sqrt{3}+2\sqrt{5}\sqrt{2}+2\sqrt{5}\sqrt{1}+$
$2\sqrt{3}\sqrt{2}+2\sqrt{3}\sqrt{1}+2\sqrt{2}\sqrt{1}.$

$= 17+\sqrt{120}+\sqrt{72}+\sqrt{48}+\sqrt{24}+\sqrt{60}+\sqrt{40}+\sqrt{20}+\sqrt{24}+$
$\sqrt{12}+\sqrt{8}.$
$= 17+\sqrt{120}+\sqrt{72}+\sqrt{60}+\sqrt{48}+\sqrt{40}+\sqrt{24}+\sqrt{24}+\sqrt{20}+\sqrt{12}+\sqrt{8}.$

(v). $(\sqrt{32}+\sqrt{18}+\sqrt{8}+\sqrt{2})^2$

As already stated in verse 31, and later in the commentary under verse 38, multiplication, division, squaring or extraction of square-root of the surd expression, should proceed with after addition of two or more terms of the multiplier and multiplicand or of the divisor and dividend where ever it is possible.

Here, $\sqrt{32}+\sqrt{18}+\sqrt{8}+\sqrt{2} = 4\sqrt{2}+3\sqrt{2}+2\sqrt{2}+1\sqrt{2}$
$= 10\sqrt{2}$
$= \sqrt{200}.$

$$\therefore (\sqrt{32}+\sqrt{18}+\sqrt{8}+\sqrt{2})^2 = (\sqrt{200})^2$$
$$= 200.$$

Following examples illustrate the maxim stated in verses 39 and 40.

उदाहरणम् –

> पञ्चत्रिमितकरण्योरन्तरवर्गं वदाशु मे विद्वन् ।
> पञ्चत्रिद्विमितानां स्वस्वर्णानामृणधनगानाम् ॥२०॥

Ex. 20 : **"**Oh scholar **!** Tell me the square of the difference of $\sqrt{5}$ and $\sqrt{3}$; and that of $(\sqrt{5}+\sqrt{3}-\sqrt{2})$, and also by making the positive negative and the negative positive in it." II **20** II

Statement : Given surds : $\sqrt{5}$ and $\sqrt{3}$.

Difference of the given surds : $(\sqrt{5}-\sqrt{3})$ *or* $(-\sqrt{5}+\sqrt{3})$.

(i). $(\sqrt{5}-\sqrt{3})^2 = 5+3+2(\sqrt{5})\cdot(-\sqrt{3})\cdot$
$= 8-2\sqrt{15}$
$= 8-\sqrt{60}$.

Or

$(-\sqrt{5}+\sqrt{3})^2 = 5+3+2(-\sqrt{5})\cdot(\sqrt{3})\cdot$
$= 8-2\sqrt{15}$
$= 8-\sqrt{60}$.

(ii). $(\sqrt{5}+\sqrt{3}-\sqrt{2})^2 = 5+3+2+2(\sqrt{5})\cdot(\sqrt{3})$
$+2(\sqrt{3})\cdot(-\sqrt{2})+2(-\sqrt{2})\cdot(\sqrt{5})$.
$= 10+\sqrt{60}-\sqrt{24}-\sqrt{40}.$

Or

$(-\sqrt{5}-\sqrt{3}+\sqrt{2})^2 = 5+3+2+2(-\sqrt{5})\cdot(-\sqrt{3})$
$+2(-\sqrt{3})\cdot(\sqrt{2})+2(\sqrt{2})\cdot(-\sqrt{5})$.
$= 10+\sqrt{60}-\sqrt{24}-\sqrt{40}.$

इतिकरणीवर्गः ।

Thus ends the operation of squaring of the surds.

5.6.1: Method of Remainders : [Rules for extracting squre-root of a surd expression (and the number of component surd-numbers to be therei'n)]:

करणीवर्गमूले सूत्रम् (शेषविधिः)–

करणीवर्गे नियमः सङ्कलितमितानि (खण्डकानि) स्युः।
एककरण्या वर्गे रूपाण्येव द्वयोः सरूप च ॥४१॥

तिसृणां तिस्रश्च तथा षडपि चतुर्णां दशपि पञ्चानाम् ।
षण्णामपि पञ्चदशेत्येवं ज्ञेयानि खण्डानि ॥४२॥

सङ्कलितात्मकमूले तन्मितखण्डैक्यतुल्यरूपाणि ।
रूपकृतेः प्रोज्झ्य पदं तेनोनयुतानि रूपाणि ॥४३॥

दलिते करणीखण्डे सन्ति करण्यः कृतौ शेषाः ।
महती रूपणि तयोः प्राग्वत्साध्येऽपरे खण्डे ॥४४॥

सङ्कलितात्मकमूलाभावे खण्डेषु तेषु खण्डानि ।
विश्लेष्य यथा मूलप्राप्तिः स्यादन्यथैवासत् ॥४५॥

"The number of irrational terms in the square of a surd expression is equal to the sum of natural numbers : this is the usual rule. In the square of a single surd term, there is only a rational number. In the square of an expression consisting of two surd terms, there is one surd term together with a rational number; of three ; three ; of four ; six ; of five, ten ; and in the square of an expression consisting of six surd terms, there will be as many as fifteen surd terms ; so it should be known.[10] ॥ **41-42** ॥

[10] .Cf. BBi. vs.44-45(i). . IJHS, **28**(3), 1993, p. 260.

In an expression having the number of surd terms equal to the sum of the natural numbers, subtract from the square of the rational term a rational number equal to the sum of that number of surd numbers and then extract the square root of the remainder. Add and subtract this to the rational number and halve. The results are the two surd terms. If further terms remain to be operated upon, regard the greater of these two as a rational number and find the other terms (of the root) by poceeding as before.[11] ॥ **43 - 44** ॥

If the number of surd terms in any expression be not equal to the sum of the natural numbers, the (requisite) number should be made up by breaking up some of the terms and then the square-root should be extracted. If that is not possible, the problem is wrong."[12] ॥ **45** ॥

Prior to Nārāyaṇa, Bhāskara II has stated the above rules, and also added the following explanatory notes[13]:

करणीवर्गराशौ रूपैरवश्यं भवितव्यम् । एककरण्या वर्गे रूपाण्येव, द्वयोः सरूपैका करणी तिसृणां तिस्रः, चतसृणां षट् । पञ्चानां दश । षण्णां पञ्चदश इत्यादि ।

अतो द्व्यादीनां करणीनां वर्णेषु एकादिसंकलितमितानि करणीनां खण्डानि रूपाणि च यथाक्रमं स्युः । अथ यदि उदाहरणे तावन्ति न

[11] . (i).BrSpSi. xviii-40; (ii).SiSe. xiv-12. (iii). BBi.॥४१॥ 39–40 ॥ Cf. IJHS, **28**(3), 1993, pp. 258-259,f.n.26.

[12] . Cf. BBi. R. ॥४४/44-45॥. IJHS, **28**(3), 1993, pp. 262-263,f.n.30.

[13] . *IJHS.* **28** (3), 1993. p. 260-261.

भवन्ति तदाऽसौ योगकरणी विश्लेष्या वा भवतीति कृत्वा मूलं ग्राह्यमित्यर्थः ।

"In the square of an expression containing irrational terms, there must be a rational term. In the square of (an expression consisting of) a single surd, there will be only a rational term; of two surds, one surd together with a rational term; of three surds, three irrational terms and a rational term; of four surds. six; of five surds, ten; of six surds, fifteen; and so on.

Thus, in the square of surd expressions consisting of two or more irrational terms, the numbers of irrational terms will be equal to the sum of the natural numbers one etc. respectively, besides the rational term. So if in an example (proposed), the number (of irrational terms present) be not such; then it must be considered as a compound surd. Break it up (into required number of component surds) and then extract the square-root. This is what has been implied."

5.7: Method of Quotients or (Nārāyaṇa's Method) [Alternative-Rule for extracting squre-root of a surd expression]:

अथ प्रकारान्तरमाह (लब्धिः विधानं)–

अथवा सर्वकरण्यश्चतुर्विभक्ता न्यसेदनल्पायाः ।
आद्यासन्नकरण्योर्हतिराद्याप्ता च तत्पदं करणी ॥४६॥

ते एव तया भक्ते करणीखण्डे परे भवतः ।
क्रमशस्तैरपि खण्डैः शेषाः भक्ताः पराकरणी ॥४७॥

तैरपि मुहुः सलब्धैः शेषः करणीर्भजेत् प्राग्वत् ।
पदकरणीवर्गयुतिं विशोधयेद् रूपवर्गेभ्यः ॥४८॥

एवं कृते तु न यदा तदा भवेद्योगजा करणी ।
विश्लेषसमुत्पाद्याः करणीवर्गे करण्योऽन्याः ॥४९॥

"Or divide all the surd numbers (present in an expression) by four and arrange the quotients in the descending order. Divide the product of the two surds nearest to the first surd (in the series) by the latter. The square root of the quotient will be a surd term (in the root).

Those two surds divided by this root will give another two surd terms (of the root).

By these (three surds) divide next (three) terms of the series and the quotient will be another surd of the root.

Again by these should be divided the other terms and the quotient is another surd; and so on.

If now the square of the sum of surd numbers (in the root) be subtracted from the square of the rational term (in the given expression) no remainder will be left.

If it be not so (i.e., if a remainder is left), then the (given) square expression is a compound surd and it should be broken up into other surds by the rule of separation."[14]॥ **46-49** ॥

5.8 : Rule prescribing the number of irrational terms (to be subtracted from the square of the rational term), and their order (while extracting the square-root):

करणीखण्डमितिर्या द्विगुणा रूपांघ्रियुक्तया मूलम् ।
रूपदलेन $\frac{1}{2}$ विहीनं सङ्कलितपदं भवत्येव ॥५०॥

" Increase twice the number of surd terms (in a given expression) by one fourth and then extract the square-root. Subtract half from that. The residue will give the number of terms (the sum of which is to be subtracted from the square of the rational term."[15] ॥ **50** ॥

[14] . IJHS, **28**(3), 1993, p. 263, f.n.38.

[15] . Cf. BBi. R. ॥४४/44-45॥. IJHS, **28**(3), 1993, p. -263, f.n.37.

5.9: Illustrtive Examples for Method of Remainders : (i.e., for rule stated in vs.50 and 41-45):

(5.9.1). Find the square-root of $5 + \sqrt{24}$.

Solution :

Here there is only one surd term in the given expression.

In accordance with the rule stated in verse-50 :

(i). Double the number of surd terms : $1 \times 2 = 2$,

(ii). Add $(1/4)$ to it : $2 + (1/4) = (9/4)$,

(iii). Extract the square-root of the sum : $\sqrt{(9/4)} = (3/2)$,

(iv). Subtract $\frac{1}{2}$ from that : $\frac{3}{2} - \frac{1}{2} = 1$,

(v). The residue is the number of surd terms which are to be subtracted from the square of the rational term. In this case it is 1.

Then, proceeding as per the rule stated in verses 43-44 :

(vi). Subtracting the (only) surd-number 24 inthe given expression from the square of the rational term in it : we have $5^2 - 24 = 1$,

(vii). Square-root of the remainder : $\sqrt{1} = 1$,

(viii). Adding this to, and subtracting from the rational term : $5 + 1 = 6,\ 5 - 1 = 4$;

(ix). Taking their moieties (halves) : $\frac{6}{2} = 3,\ \frac{4}{2} = 2$;

(x). These are the two component surd-numbers in the square-root of the given expression :

$$\therefore \sqrt{5 + \sqrt{24}} = \sqrt{3} + \sqrt{2}\,.$$

(5.9.2). Find the square-root of $11 + \sqrt{72} + \sqrt{48} + \sqrt{24}$.

Solution :

Here there are three surd terms in the given expression.

In accordance with the rule stated in verse-50 :

(i). Double the number of surd terms : $3 \times 2 = 6$,

(ii). Add $(1/4)$ to it : $6 + (1/4) = (25/4)$,

(iii). Extract the square-root of the sum : $\sqrt{(25/4)} = (5/2)$,

(iv). Subtract $\frac{1}{2}$ from that : $\frac{5}{2} - \frac{1}{2} = 2$,

(v). The residue is the number of surd terms which are to be subtracted from the square of the rational term. In this case it is 2.

Then, proceeding as per the rule stated in verses 43-44 :

(vi). Subtracting the two surd-numbers 48 and 24 inthe given expression from the square of the rational term : we have $121 - (48 + 24) = 49$,

(vii). Square-root of the remainder : $\sqrt{49} = 7$,

(viii). Adding this to, and subtracting from the rational term : $11 + 7 = 18,\ 11 - 7 = 4$;

(ix). Taking their moieties (halves) : $\frac{18}{2} = 9,\ \frac{4}{2} = \mathbf{2}$;

These are the two component surd-numbers in the square-root of the given expression. Now , regard 9, the greater of these two as a rational number, and subtracting the remaining one surd number 72 from the square of the rational term find the other terms (of the root) by poceeding as before : That is,

(x). $\frac{1}{2}\{9 \pm \sqrt{9^2 - 72}\} = (\mathbf{6},\ \mathbf{3})$;

Thus the surd numbers obtained are : $\sqrt{6}, \sqrt{3}$. and $\sqrt{2}$.

$$\therefore \sqrt{(11 + \sqrt{72} + \sqrt{48} + \sqrt{24})} = \sqrt{6} + \sqrt{3} + \sqrt{2}\,.$$

Or

Afterworking out the first five steps as shown above, and then, proceeding as per the rule stated in verses 43-44 :

(vi). Subtracting the two surd-numbers 72 and 24 inthe given expression from the square of the rational term :
we have $121-(72+24) = 25$,

(vii). Square-root of the remainder : $\sqrt{25} = 5$,

(viii). Adding this to, and subtracting from the rational term :
$11+5 = 16,\ 11-5 = 6$;

(ix). Taking their moieties (halves) : $\frac{16}{2} = 8,\ \frac{6}{2} = \mathbf{3}$;

These are the two component surd-numbers in the square-root of the given expression. Now , regard 8, the greater of these two as a rational number and subtracting the remaining one surd number 48 from the square of the rational term find the other terms (of the root) by poceeding as before : That is,

(x). $\frac{1}{2}\{8 \pm \sqrt{8^2-48}\} = (\mathbf{6},\ \mathbf{2})$;

Thus the surd numbers of the root are : $\sqrt{6}, \sqrt{3}.$ and $\sqrt{2}$.

$$\therefore \sqrt{(11+\sqrt{72}+\ \sqrt{48}+\sqrt{24}\,)} = \sqrt{6}+\sqrt{3}+\sqrt{2}\,.$$

Or

Afterworking oout the first five steps as shown above, and then, proceeding as per the rule stated in verses 43-44 :

(vi). Subtracting the two surd-numbers 72 and 48 inthe given expression from the square of the rational term :
we have $121-(72+48) = 1$,

(vii). Square-root of the remainder : $\sqrt{1} = 1$,

(viii). Adding this to, and subtracting from the rational term :
$11+1 = 12,\ 11-1 = 10$;

(ix). Taking their moieties (halves) : $\frac{12}{2} = \mathbf{6},\ \frac{10}{2} = 5$;

These are the two component surd-numbers in the square-root of the given expression. Now , regard 5, the smaller of these two as a rational number and and subtracting the remaining one surd number 24 from the square of the rational term find the other terms (of the root) by poceeding as before : That is,

(x). $\frac{1}{2}\{5 \pm \sqrt{5^2-24}\} = (\mathbf{3},\ \mathbf{2})$;

Thus the surd numbers obtained in the root of the given expression are : $\sqrt{6}, \sqrt{3}.$ and $\sqrt{2}$.

$$\therefore \sqrt{(11+\sqrt{72}+\ \sqrt{48}+\sqrt{24}\,)} = \sqrt{6}+\sqrt{3}+\sqrt{2}\,.$$

(5.9.3). Find the square-root of :

$$\mathbf{17}+\sqrt{\mathbf{140}}+\sqrt{\mathbf{84}}+\sqrt{\mathbf{60}}\ \mathbf{++}\sqrt{\mathbf{56}}+\sqrt{\mathbf{40}}+\sqrt{\mathbf{24}}\,\mathbf{.}$$

Solution :

Here there are six surd terms in the given expression. In accordance with the rule stated in verse-50 :

(i). Double the number of surd terms : $6\times 2 = 12$,

(ii). Add $(1/4)$ to it : $12+(1/4) = (49/4)$,

(iii). Extract the square-root of the sum : $\sqrt{(49/4)} = (7/2)$,

(iv). Subtract $\frac{1}{2}$ from that : $\frac{7}{2}-\frac{1}{2} = 3$,

(v). The residue is the number of surd terms which are to be subtracted from the square of the rational term. In this case it is 3.

Then, proceeding as per the rule stated in verses 43-44 :

(vi). Subtracting the three surd-numbers 56, 40, and 24 inthe given expression from the square of the rational term : we have $289 - (56 + 40 + 24) = 169$,

(vii). Square-root of the remainder : $\sqrt{169} = 13$,

(viii). Adding this to, and subtracting from the rational term : $17 + 13 = 30,\ 17 - 13 = 4$;

(ix). Taking their moieties (halves) : $\frac{30}{2} = 15,\ \frac{4}{2} = \mathbf{2}$;

These are the two component surd-numbers in the square-root of the given expression. Now , regard15, the greater of these two as a rational number and and subtracting the two surd numbers 84 and 60 among the remaining, from the square of the rational term find the other terms (of the root) by poceeding as before : That is,

(x). $\frac{1}{2}\left\{15 \pm \sqrt{15^2 - (84 + 60)}\right\} = (12,\ \mathbf{3})$;

Again, regard12, the greater of these two as a rational number and subtracting the only remaining surd number 140 from the square of the rational term and find the other terms (of the root) by poceeding as before : That is,

$\frac{1}{2}\left\{12 \pm \sqrt{12^2 - (140)}\right\} = (\mathbf{7},\ \mathbf{5})$.

Thus the surd numbers obtained are : $\sqrt{7},\ \sqrt{5},\ \sqrt{3}.$ and $\sqrt{2}$.

$$\therefore \sqrt{(17 + \sqrt{140} + \sqrt{84} + \sqrt{60} + + \sqrt{56} + \sqrt{40} + \sqrt{24})}$$
$$= (\sqrt{7} + \sqrt{5} + \sqrt{3} + \sqrt{2}).$$

(5.9.4). Find the square-root of :

$$\mathbf{17 + \sqrt{128} + \sqrt{120} + \sqrt{108} + + \sqrt{96} + \sqrt{60} + \sqrt{40} + \sqrt{20}.}$$

Solution :

Here, the number of component surd terms in the given expression is 7. Proceeding as per the rule stated in verse 50 :

Multiplying by 2 and adding $(1/4)$ to the product we get, $(7 \times 2) + (1/4) = (57/4)$. This is a non-square number and do not have an integral root. There fore, the the number (of irrational terms present in the given expression is not equal to (सङ्कलितपद) the number in the series of the sum of the natural numbers, so the given expression is a compound surd. Hence, break it up into required number of component surds and then extract the square-root.

As $\sqrt{128} = 4\sqrt{8} = 3\sqrt{8} + 1.\sqrt{8}$; break the term$\sqrt{128}$ into the two surd term $\sqrt{72} + \sqrt{8}$. Now the number of component surds is 8, this also is not equal to (सङ्कलितपद) the number in the series of the sum of the natural numbers .

Now [as $\sqrt{96} = 2\sqrt{24} = \sqrt{24} + \sqrt{24}$] break the term$\sqrt{96}$ into the two surd term $\sqrt{24} + \sqrt{24}$. Now the number of component surds is 9. Again, this also is not equal to (सङ्कलितपद) the number in the series of the sum of the natural numbers .

Now [as $\sqrt{108} = 6\sqrt{3} = 4\sqrt{3} + 2\sqrt{3}$] break the term$\sqrt{108}$ into the two surd term $\sqrt{48} + \sqrt{12}$. Now the number of

component surds is 10. This is equal to (सङ्कलितपद,) the number in the series of the sum of the natural numbers .

So, rewriting the given expression, it will be as follows:

$$17+\sqrt{120}+\sqrt{72}+\sqrt{60}+\sqrt{48}+\sqrt{40}+\sqrt{24}+\sqrt{24}+\sqrt{20}+\sqrt{12}+\sqrt{8}.$$

Now proceeding as per the rule stated in verses 43-44 :

(i). Subtracting the four surd-numbers 24, 20, 12 and 8 inthe given expression from the square of the rational term : we have $289-(24+20+12+8)=225$,

(ii). Square-root of the remainder : $\sqrt{225}=15$,

(iii). Adding this to, and subtracting from the rational term : $17+15=32,\ 17-15=2$;

(iv). Taking their moieties (halves) : $\frac{32}{2}=16,\ \frac{2}{2}=\mathbf{1}$;

These are the two component surd-numbers in the square-root of the given expression.

(v). Now , regard16, the greater of these two as a rational number and and subtracting the three surd numbers 48, 40, and 24 among the remaining, from the square of the rational term find the other terms (of the root) by poceeding as before : That is,

$$\frac{1}{2}\left\{16\pm\sqrt{16^2-(48+40+24)}\right\}=(14,\ \mathbf{2});$$

(vi). Again, regard14, the greater of these two as a rational number and Subtracting the two surd number 72 and 60, among the remaining three surds, from the square of the rational term and find the other terms (of the root) by poceeding as before : That is,

$$\frac{1}{2}\left\{14\pm\sqrt{14^2-(72+60)}\right\}=(11,\ \mathbf{3}).$$

(vii) Again, regard11, the greater of these two as a rational number and subtracting the only surd number left 120 from the square of the rational term and find the other terms (of the root) by poceeding as before : That is,

$$\frac{1}{2}\left\{11\pm\sqrt{11^2-(120)}\right\}=(\mathbf{6},\ \mathbf{5}).$$

Thus the surd numbers obtained are : $\sqrt{6},\ \sqrt{5},\ \sqrt{3}.\ \sqrt{2}$and $\sqrt{1}$.

$$\therefore\ \sqrt{\left(17+\sqrt{128}+\sqrt{120}+\sqrt{108}++\sqrt{96}+\sqrt{60}+\sqrt{40}+\sqrt{20}\right)}$$
$$=\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}+\sqrt{1}\ .$$

(5.9.5). Find the square-root of :

$$\mathbf{60}+\sqrt{\mathbf{2304}}+\sqrt{\mathbf{1024}}+\sqrt{\mathbf{576}}+\sqrt{\mathbf{256}}+\sqrt{\mathbf{144}}+\sqrt{\mathbf{64}}\,.$$

Here there are six surd terms in the given expression. Now proceeding as per the rule stated in verses 43-44 :

(i). Subtracting the three surd-numbers 256, 144, and 64 in the given expression from the square of the rational term in it : we have $3600-(256+144+64)=3136$,

(ii). Square-root of the remainder : $\sqrt{3136}=56$,

(iii). Adding this to, and subtracting from the rational term : $60+56=116,\ 60-56=4$;

Taking their moieties (halves) : $\frac{116}{2}=58,\ \frac{4}{2}=\mathbf{2}$;

These are the two component surd-numbers in the square-root of the given expression.

(iv). Now , regard58, the greater of these two as a rational number and and subtracting the two surd numbers 1024 and 576 (among the remainingthree,) from the square of the rational term find the other terms of the root by poceeding as before : That is,

$$\frac{1}{2}\left\{58 \pm \sqrt{58^2 - (1024 + 576)}\right\} = (50,\ \mathbf{8});$$

(v). Again, regard 50, the greater of these two as a rational number and subtracting the only remaining surd number 2304 from the square of the rational term and find the other terms (of the root) by poceeding as before : That is,

$$\frac{1}{2}\left\{50 \pm \sqrt{50^2 - (2304)}\right\} = (\mathbf{32},\ \mathbf{18}).$$

Thus the surd numbers obtained are : 32, 18, 8, 2.

$$\therefore \sqrt{(60 + \sqrt{2304} + \sqrt{1024} + \sqrt{576} + \sqrt{256} + \sqrt{144} + \sqrt{64}\,)}$$
$$= \sqrt{32} + \sqrt{18} + \sqrt{8} + \sqrt{2}\,.$$

Or

The sum of the roots of all the component surds including the rational term is equal to :

$$60 + (48 + 32 + 24 + 16 + 12 + 8) = 200.$$

Now according to the definition (stated in the first half of the verse 25, "the number whose square-root is to be taken is *karaṇī*."

$$\therefore \sqrt{(60 + \sqrt{2304} + \sqrt{1024} + \sqrt{576} + \sqrt{256} + \sqrt{144} + \sqrt{64}\,)}$$
$$= \sqrt{60 + (48 + 32 + 24 + 16 + 12 + 8)} \quad = \sqrt{200}.$$

By the rule of separation stated in verse 36,

$$\sqrt{200} = \sqrt{32} + \sqrt{18} + \sqrt{8} + \sqrt{2}\,.$$

5.10 : Illustrative Examples for Method of quotients :
[i.e., Alternative-Rule stated in verses 46-49 for extracting squre-root of a surd expression]:

(5.10.1). Find the square-root of $5 + \sqrt{24}$.

As there is only one term in the surd expression, the alternative rule for the extraction of the square-root can not be applied here. Hence another method is followed.

Now Let, $\mathbf{5} + \sqrt{\mathbf{24}} = \{(\boldsymbol{a} + \boldsymbol{b}) + \mathbf{2}\sqrt{\boldsymbol{ab}}\} = (\sqrt{\boldsymbol{a}} + \sqrt{\boldsymbol{b}})^{\mathbf{2}}$.
then the required result is to find $\sqrt{a} + \sqrt{b}$.

Here, $(a + b) = 5$; $2\sqrt{ab} = \sqrt{24}$; $\therefore ab = \frac{24}{4} = 6$

But $(a - b) = \sqrt{(a-b)^2} = \sqrt{\{(a + b)^2 - 4ab\}}$

$\therefore\ (a - b) = \sqrt{5^2 - 4 \times 6} = 1.$

Then by the rule of concurrence (*saṅkramaṇa*):

$$(a,\qquad b) = (1/2)\{\,5\ \pm\ 1\,\} = (3,\qquad 2)$$

$$\therefore \sqrt{(\mathbf{5} + \sqrt{\mathbf{24}})} = \sqrt{a} + \sqrt{b} = \sqrt{3} + \sqrt{2}\,.$$

(5.10.2). Find the square-root of : $\mathbf{11} + \sqrt{\mathbf{72}} + \sqrt{\mathbf{48}} + \sqrt{\mathbf{24}}$.

Solution :

(i). Divide all the surd numbers (present in an expression) by four and arrange the quotients in the descending order : $\sqrt{(72/4)},\ \sqrt{(48/4)},\ \sqrt{(24/4)}$.

That is, $\sqrt{18},\ \sqrt{12},\ \sqrt{6}$.

(ii). Divide the product of the two surds nearest to the first surd (in the series) by the latter : $\left\{\frac{\sqrt{12}\times\sqrt{6}}{\sqrt{18}}\right\} = \sqrt{4} = 2$.
The square root of the quotient , i.e., $\sqrt{\mathbf{2}}$ will be a surd term (in the root).

(iii). Those two surds divided by this root will give another two surd terms (of the root) : $\frac{\sqrt{12}}{\sqrt{2}} = \sqrt{\mathbf{6}}$, $\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{\mathbf{3}}$ **.**
Thus the three surd terms obtained are : $\sqrt{6}$, $\sqrt{3}$, $\sqrt{2}$.

(iv) . Now these will be the surd terms in the root of the given expression, if the square of the sum of the surd numbers when subtracted from the square of the rational term, leaves no remainder. $\{11^2 - (6 + 3 + 2)^2\} = 0$.

(v). $\therefore \sqrt{\mathbf{11} + \sqrt{\mathbf{72}} + \sqrt{\mathbf{48}} + \sqrt{\mathbf{24}}} = \sqrt{6} + \sqrt{3} + \sqrt{2}$.

(5.10.3). Find the square-root of :

$$\mathbf{17} + \sqrt{\mathbf{140}} + \sqrt{\mathbf{84}} + \sqrt{\mathbf{60}} \ \mathbf{++} \sqrt{\mathbf{56}} + \sqrt{\mathbf{40}} + \sqrt{\mathbf{24}}\ \mathbf{.}$$

Solution :

(i). Divide all the all the surd numbers (present in an expression) by four and arrange the quotients in the descending order : $\sqrt{(140/4)}$, $\sqrt{(84/4)}$, $\sqrt{(60/4)}$, $\sqrt{(56/4)}$, $\sqrt{(40/4)}$, $\sqrt{(24/4)}$.

That is, $\sqrt{35}$, $\sqrt{21}$, $\sqrt{15}$, $\sqrt{14}$, $\sqrt{10}$, $\sqrt{6}$.

(ii). Divide the product of the two surds nearest to the first surd (in the series) by the latter : $\left\{\frac{\sqrt{21}\times\sqrt{15}}{\sqrt{35}}\right\} = \sqrt{9} = 3$.
The square root of the quotient , i.e., $\sqrt{\mathbf{3}}$ will be a surd term (in the root).

(iii). Those two surds divided by this root will give another two surd terms (of the root) : $\frac{\sqrt{21}}{\sqrt{3}} = \sqrt{\mathbf{7}}$, $\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{\mathbf{5}}$ **.**
Thus the three surd terms obtained are : $\sqrt{7}$, $\sqrt{5}$, $\sqrt{3}$.

(iv). By these three surd terms divide the next three terms and the quotient will be another surd of the root:
That is, $\frac{\sqrt{14}}{\sqrt{7}} = \sqrt{2}$, $\frac{\sqrt{10}}{\sqrt{5}} = \sqrt{2}$, $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$.
Thus the four surd terms obtained are : $\sqrt{7}$, $\sqrt{5}$, $\sqrt{3}$, $\sqrt{2}$.

(iv) . Now these will be the surd terms in the root of the given expression, if the square of the sum of the surd numbers when subtracted from the square of the rational term, leaves no remainder $\{17^2 - (7 + 5 + 3 + 2)^2\} = 0$.

(v). $\therefore \sqrt{(17 + \sqrt{140} + \sqrt{84} + \sqrt{60} + + \sqrt{56} + \sqrt{40} + \sqrt{24})}$
$= \sqrt{7} + \sqrt{5} + \sqrt{3} + \sqrt{2}$.

(5.10.4). Find the square-root of :

$$\mathbf{17} + \sqrt{\mathbf{128}} + \sqrt{\mathbf{120}} + \sqrt{\mathbf{108}} + + \sqrt{\mathbf{96}} + \sqrt{\mathbf{60}} + \sqrt{\mathbf{40}} + \sqrt{\mathbf{20}}.$$

Solution :

The given expression is a compound surd. Rrewriting the given expression, with the number of terms of equal to (सङ्कलितपद,) the number in the series of the sum of the natural numbers, it will be as follows:

$$17 + \sqrt{120} + \sqrt{72} + \sqrt{60} + \sqrt{48} + \sqrt{40} + \sqrt{24} + \sqrt{24} + \sqrt{20} + \sqrt{12} + \sqrt{8}.$$

Now proceeding as per the alternative rule for the extraction of the square-root :

(i). Divide all the all the surd numbers (present in an expression) by four and arrange the quotients in the descending order:
$\sqrt{(120/4)}$, $\sqrt{(72/4)}$, $\sqrt{(60/4)}$,$\sqrt{(48/4)}$, $\sqrt{(40/4)}$, $\sqrt{(24/4)}$, $\sqrt{(24/4)}$, $\sqrt{(20/4)}$, $\sqrt{(12/4)}$, $\sqrt{(8/4)}$.

That is, $\sqrt{30}, \sqrt{18}, \sqrt{15}, \sqrt{12}, \sqrt{10}, \sqrt{6}, \sqrt{6}, \sqrt{5}, \sqrt{3}, \sqrt{2}$.

(ii). Divide the product of the two surds nearest to the first surd (in the series) by the latter : $\left\{\frac{\sqrt{18}\times\sqrt{15}}{\sqrt{30}}\right\} = \sqrt{9} = 3$.

The square root of the quotient , i.e., $\sqrt{\mathbf{3}}$ will be a surd term (in the root).

(iii). Those two surds divided by this root will give another two surd terms (of the root) : $\frac{\sqrt{18}}{\sqrt{3}} = \sqrt{\mathbf{6}}$, $\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{\mathbf{5}}$ **.**

Thus the three surd terms obtained are : $\sqrt{6}$, $\sqrt{5}$, $\sqrt{3}$.

(iv). By these three surd terms divide the next three terms and the quotient will be another surd of the root:

That is, $\frac{\sqrt{12}}{\sqrt{6}} = \sqrt{2}$, $\frac{\sqrt{10}}{\sqrt{5}} = \sqrt{2}$, $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$

(v). By these four surd terms divide the next four terms and the quotient will be another surd of the root:

That is, $\frac{\sqrt{6}}{\sqrt{6}} = \sqrt{1}$, $\frac{\sqrt{5}}{\sqrt{5}} = \sqrt{1}$, $\frac{\sqrt{3}}{\sqrt{3}} = \sqrt{1}$, $\frac{\sqrt{2}}{\sqrt{2}} = \sqrt{1}$.

Thus the five surd terms obtained are :

(vi) . Now these will be the surd terms in th$\sqrt{6}, \sqrt{5}, \sqrt{3}, \sqrt{2}\ \sqrt{1}$.e root of the given expression, if the

square of the sum of the surd numbers when subtracted from the square of the rational term, leaves no remainder :

$$\{17^2 - (6 + 5 + 3 + 2 + 1)^2\} = 0.$$

$$\therefore \sqrt{\left(17 + \sqrt{128} + \sqrt{120} + \sqrt{108} + + \sqrt{96} + \sqrt{60} + \sqrt{40} + \sqrt{20}\right)}$$
$$= \sqrt{6} + \sqrt{5} + \sqrt{3} + \sqrt{2} + \sqrt{1} .$$

(5.10.5). Find the square-root of :

$$\mathbf{60} + \sqrt{\mathbf{2304}} + \sqrt{\mathbf{1024}} + \sqrt{\mathbf{576}} + \sqrt{\mathbf{256}} + \sqrt{\mathbf{144}} + \sqrt{\mathbf{64}} .$$

Solution :

(i). Divide all the all the surd numbers (present in an expression) by four and arrange the quotients in the descending order : $\sqrt{(2304/4)}$, $\sqrt{(1024/4)}$, $\sqrt{(576/4)}$, $\sqrt{(256/4)}$, $\sqrt{(144/4)}$, $\sqrt{(64/4)}$

That is, $\sqrt{576}$, $\sqrt{256}$, $\sqrt{144}$, $\sqrt{64}$, $\sqrt{36}$, $\sqrt{16}$, .

(ii). Divide the product of the two surds nearest to the first surd (in the series) by the latter : $\left\{\frac{\sqrt{256}\times\sqrt{144}}{\sqrt{576}}\right\} = \sqrt{64} = 8$.

The square root of the quotient , i.e., $\sqrt{\mathbf{8}}$ will be a surd term (in the root).

(iii). Those two surds divided by this root will give another two surd terms (of the root) : $\frac{\sqrt{256}}{\sqrt{8}} = \sqrt{\mathbf{32}}$, $\frac{\sqrt{144}}{\sqrt{8}} = \sqrt{\mathbf{18}}$ **.**

Thus the three surd terms obtained are: $\sqrt{32}$, $\sqrt{18}$, $\sqrt{8}$.

(iv). By these three surd terms divide the next three terms and the quotient will be another surd of the root:

That is, $\frac{\sqrt{64}}{\sqrt{32}} = \sqrt{2}$, $\frac{\sqrt{36}}{\sqrt{18}} = \sqrt{2}$, $\frac{\sqrt{16}}{\sqrt{8}} = \sqrt{2}$

Thus the four surd terms obtained are : $\sqrt{32}$, $\sqrt{18}$, $\sqrt{8}, \sqrt{2}$.

(iv) . Now these will be the surd terms in the root of the given expression, if the square of the sum of the surd numbers when subtracted from the square of the rational term, leaves no remainder $\{60^2 - (32 + 18 + 8 + 2)^2\} = 0$.

(v). $\therefore \sqrt{(60 + \sqrt{2304} + \sqrt{1024} + \sqrt{576} + \sqrt{256} + \sqrt{144} + \sqrt{64})}$

$= \sqrt{32} + \sqrt{18} + \sqrt{8} + \sqrt{2}$.

अपि च –

प्रकृतिपुरन्दरनगरसगुणभुजतुल्याश्चतुर्गुणा विद्वन् ।
वर्गे यत्र करण्यः सविश्वरूपाः पदं ब्रूहि ॥२१॥

Ex. 21 : "Oh wise-man ! Where The component surd numbers in the square of a surd-expression are : four times (21, 14, 7, 6, 3, 2) and the rational number is 13. Tell (me) the square-root (of that)." ॥ **21** ॥

***Nyāsa* (Statement)** : Find the square-root of :

$\left[13 + \sqrt{(4 \times 21)} + \sqrt{(4 \times 14)} + \sqrt{(4 \times 7)} + \sqrt{(4 \times 6)} + \sqrt{(4 \times 3)} + \sqrt{(4 \times 2)}\right]$.

That is, $\{13 + \sqrt{84} + \sqrt{56} + \sqrt{28} + \sqrt{24} + \sqrt{12} + \sqrt{8}\}$.

Solution :

Now proceeding as per the alternative rule (stated inverses 46-49) for the extraction of the square-root :

(i). Divide all the all the surd numbers (present in an expression) by four and arrange the quotients in the descending order: $\frac{\sqrt{84}}{\sqrt{4}} = \sqrt{21}$, $\frac{\sqrt{56}}{\sqrt{4}} = \sqrt{14}$, $\frac{\sqrt{28}}{\sqrt{4}} = \sqrt{7}$, $\frac{\sqrt{24}}{\sqrt{4}} = \sqrt{6}$, $\frac{\sqrt{12}}{\sqrt{4}} = \sqrt{3}$, $\frac{\sqrt{8}}{\sqrt{4}} = \sqrt{2}$.

(ii). Divide the product of the two surds nearest to the first surd (in the series) by the latter : $\left\{\frac{\sqrt{14} \times \sqrt{7}}{\sqrt{21}}\right\} = \frac{\sqrt{98}}{\sqrt{21}}$. As this is a non-square number, integral root of it does not exist. So, divide the product of $\sqrt{21}$ and $\sqrt{6}$ by $\sqrt{14}$: $\frac{\sqrt{21} \times \sqrt{6}}{\sqrt{14}} = \sqrt{9}$.

The square root of the quotient , i.e., $\mathbf{\sqrt{3}}$ will be a surd term (in the root).

(iii). Those two surds divided by this root will give another two surd terms (of the root) : $\frac{\sqrt{21}}{\sqrt{3}} = \mathbf{\sqrt{7}}$, $\frac{\sqrt{6}}{\sqrt{3}} = \mathbf{\sqrt{2}}$ **.**

Thus the three surd terms obtained are : $\sqrt{7}$, $\sqrt{3}$, $\sqrt{2}$.

(iv). By these three surd terms divide the next three terms and the quotient will be another surd of the root:

That is, $\frac{\sqrt{7}}{\sqrt{7}} = \sqrt{1}$, $\frac{\sqrt{3}}{\sqrt{3}} = \sqrt{1}$, $\frac{\sqrt{2}}{\sqrt{2}} = \sqrt{1}$.

Thus the four surd terms obtained are : $\sqrt{7}$, $\sqrt{3}, \sqrt{2}$, $\sqrt{1}$.

(iv) . Now these will be the surd terms in the root of the given expression, if the square of the sum of the surd numbers when subtracted from the square of the rational term, leaves no remainder $\{13^2 - (7 + 3 + 2 + 1)^2\} = 0$.

(v). $\therefore \sqrt{(13 + \sqrt{84} + \sqrt{56} + \sqrt{28} + \sqrt{24} + \sqrt{12} + \sqrt{8})}$

$= \sqrt{7} + \sqrt{3} + \sqrt{2} + \sqrt{1}$.

Or

After the first step,

(ii). Divide the product of $\sqrt{21}$ and $\sqrt{3}$ by $\sqrt{7}$: $\frac{\sqrt{21}\times\sqrt{3}}{\sqrt{7}} = \sqrt{9}$

The square root of the quotient , i.e., $\sqrt{3}$ will be a surd term (in the root).

(iii). Those two surds divided by this root will give another two surd terms (of the root) : $\frac{\sqrt{21}}{\sqrt{3}} = \boldsymbol{\sqrt{7}},\ \frac{\sqrt{3}}{\sqrt{3}} = \boldsymbol{\sqrt{1}}$ **.**

Thus the three surd terms obtained are : $\sqrt{7},\ \sqrt{3},\ \sqrt{1}$.

(iv). By these three surd terms divide the next three terms and the quotient will be another surd of the root:

That is, $\frac{\sqrt{14}}{\sqrt{7}} = \sqrt{2}$, $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$, $\frac{\sqrt{2}}{\sqrt{1}} = \sqrt{2}$.

Thus the four surd terms obtained are : $\sqrt{7},\ \sqrt{3}, \sqrt{2}$, $\sqrt{1}$.

(iv) . Now these will be the surd terms in the root of the given expression, if the square of the sum of the surd numbers when subtracted from the square of the rational term, leaves no remainder $\{13^2 - (7+3+2+1)^2\} = 0$.

(v). $\therefore \sqrt{(13+\sqrt{84}+\sqrt{56}+\sqrt{28}+\sqrt{24}+\sqrt{12}+\sqrt{8})}$

$$= \sqrt{7}+\sqrt{3}+\sqrt{2}+\sqrt{1}.$$

Or

After the first step,

(ii). Divide the product of $\sqrt{14}$ and $\sqrt{2}$ by $\sqrt{7}$:

$$\frac{\sqrt{14}\times\sqrt{2}}{\sqrt{7}} = \sqrt{4} = 2$$

The square root of the quotient , i.e., $\sqrt{2}$ will be a surd term (in the root).

(iii). Those two surds divided by this root will give another two surd terms (of the root) : $\frac{\sqrt{14}}{\sqrt{2}} = \boldsymbol{\sqrt{7}},\ \frac{\sqrt{2}}{\sqrt{2}} = \boldsymbol{\sqrt{1}}$ **.**

Thus the three surd terms obtained are : $\sqrt{7},\ \sqrt{2},\ \sqrt{1}$.

(iv). By these three surd terms divide the next three terms and the quotient will be another surd of the root:

That is, $\frac{\sqrt{21}}{\sqrt{7}} = \sqrt{3}$, $\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$, $\frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$.

Thus the four surd terms obtained are : $\sqrt{7},\ \sqrt{3}, \sqrt{2}$, $\sqrt{1}$.

(iv) . Now these will be the surd terms in the root of the given expression, if the square of the sum of the surd numbers when subtracted from the square of the rational term, leaves no remainder $\{13^2 - (7+3+2+1)^2\} = 0$.

(v). $\therefore \sqrt{(13+\sqrt{84}+\sqrt{56}+\sqrt{28}+\sqrt{24}+\sqrt{12}+\sqrt{8})}$

$$= \sqrt{7}+\sqrt{3}+\sqrt{2}+\sqrt{1}.$$

5.11 : 'Divisibility Test' and 'Quotients -Identity Test'[16]

[i.e.,The nature of irrational terms that are to be subtracted] **:**

सूत्रम् –

सङ्कलितपदोत्थितयाल्पयाऽब्धिहतयाऽपवर्तनो यासाम् ।
रूपकृतेस्ता शोध्या अपवर्तफलाः करण्यः स्युः ॥५१॥
शेषविधिना न यदि तास्तदसन्मूलं तदा भवति ।

"(In extracting roots of a square-surd,) the surd-numbers that are to be subtracted from the square of the rational number should be exactly divisible by four times the smaller term in the resulting root-surd. The quotients obtained by this exact division will be the surd terms in the root. (That is, quotients in division, and component surd terms should be identical.) ॥ **51** ॥

If they are not obtained by the last rule, then the (resulting) root is wrong." ॥ **$51\frac{1}{2}$** ॥

Folllowing is an example showing that the number of surd-numbers to be subtracted from the square of the rational number, and their order, should conform to the rule stated (in verse vs.50 and 41 - 45), if not the result obtained will be wrong.

दाहरणम् –

वसुरसनेत्रप्रमिता यत्र करण्यश्चतुर्गणा वर्गे ।
युक्ता रूपैर्दशभिस्तत्र पदमं ब्रूहि मे गणक ॥२२॥

16 . Cf. BBi. ॥४४॥ 46-47 ॥ ; . *IJHS.* **28** (3), 1993. p. 260-261.

Ex. 22 : "In a surd expression, where, the square of the component surd numbers are four times (8, 6, 2), and therein the rational number being 10, O mathematician, Tell me the square-root (of that)." [17]॥ **22** ॥ That is,

Nyāsa **(Statement) :** $10 + \sqrt{(4 \times 8)} + \sqrt{(4 \times 6)} + \sqrt{(4 \times 2)}$.

Find the square-root of : $\{10 + \sqrt{32} + \sqrt{24} + \sqrt{8}\}$.

Solution :

Here, there being three surd terms, so first, a rational number equivalent to two of the surd numbers is to be subtracted from the square of the rational term and the root (of the remainder) is to be extracted; and then we have to proceed in the same way with (the remaining) one term. But it is not possible here since the remainders we get are non square numbers, and their integral-root can not be extracted. Hence this (i.e., the proposed expression) does not possess a root expressible in surd terms.

If , however, we extract the root by subtracting, contrary to the rule, an integer equivalent to all the surd terms, $i.e., (1/2)\left(10 \pm \sqrt{10^2 - 32 - 24 - 8}\right) = (2,\ 8)$.
We get $\sqrt{2} + \sqrt{8}$ as square-root of : $\{10 + \sqrt{32} + \sqrt{24} + \sqrt{8}\}$. But this is wrong as its square is 18.

Or on adding together the surds $\sqrt{32}$ and $\sqrt{8}$, (the expression becomes), $10 + \sqrt{72} + \sqrt{24}$. Then (by the rule) we obtain: $(1/2)\{10 \mp \sqrt{10^2 - 72 - 24}\} = (4, 6)$.
Therefore surd terms are $\sqrt{4} + \sqrt{6} = 2 + \sqrt{6}$.
But this is also erroneous, since its square is $10 + \sqrt{96}$.

17. BBi. Ex. 21 in verse 48.

उदाहरणम् –

तिथिमनुनयनकरण्यश्चतुर्गुणा रूपदशकसंयुक्ताः ।
किं मूलं ब्रूहि सखे करणीगणिते श्रमोऽस्ति यदि ॥२४॥

Ex. 24 : "(In a surd expression,) The surd numbers are are four times (15, 14, and 2) with the rational number 10. Oh friend ! Tell me what is the square-root (of that) if you have studied hard the mathematics of surds."[19]॥ **24** ॥

That is,

***Nyāsa* (Statement) :** .

Find the square-root of : $\{10 + \sqrt{60} + \sqrt{56} + \sqrt{8}\}$.

Solution :

Subtracting the first two terms eight and fifty-six, and proceeding as before,

$$(1/2)\left(10 \mp \sqrt{10^2 - 8 - 56}\right) = (2, 8).$$

That is, the two surd terms for the root are obtained as $\sqrt{8}$ and $\sqrt{2}$; of these the smaller one namely, 2 multiplied by four, that is 8. This exactly divides 8 and 56; and the quotient obtained are $(1,\ 7)$. But these do not come out as surds of the root [viz. (2, 8), the necessary condition laid down as stated in verse 51], by the regular process of the rule of remainder (stated inverse 39). That is, here, $(1, 7) \neq (2, 8)$ Therefore, those terms 8 and 56 are not to be subtracted. However, if violating the rule, further work is carried out,

That is, $(1/2)\left(8 \mp \sqrt{8^2 - 60}\right) = (3, 5)$.

[19] . BBi. Ex.

Then, the root obtained will be $(\sqrt{2} + \sqrt{3} + \sqrt{5})$. This is wrong, since $(\sqrt{2} + \sqrt{3} + \sqrt{5})^2 = 10 + \sqrt{24} + \sqrt{60} + \sqrt{40}$.

So, if the result is obtained without following the rules laid down, then the result will be wrong.

उदाहरणम् –

तिथिमनुरविविश्वककुभ्नगसङ्ख्याः कृतहताः करण्यश्चेत् ।
षोडशरूपसमेता यत्र कृतौ तत्र किं पदं कथय ॥२५॥

Ex. 25 : *"If* in a surd expression, where the square of the component surd numbers are four times [15, 14, 12, 13, 10, 7] with the rational number 16, Tell me, what will be the square-root there ." ॥ **25** ॥ That is,

Solution : *Nyāsa* (Statement) : $16 + \sqrt{(4 \times 15)} + \sqrt{(4 \times 14)} + \sqrt{(4 \times 12)} + \sqrt{4 \times 13} + \sqrt{4 \times 10} + \sqrt{4 \times 7}$.

Find the square-root of :

$$\{16 + \sqrt{60} + \sqrt{56} + \sqrt{48} + \sqrt{52} + \sqrt{40} + \sqrt{28}\}.$$

There being six surd terms [equal to (सङ्कलितपद)] in this, in accordance with the rule, an integer equal to three of the surd terms should be first subtracted from the square of the rational term and the root (of the remainder) found : next an integer equal to two of the surd terms and then an integer equal to one surd term (should be subtracted).

(i). $\frac{1}{2}\left\{16 \pm \sqrt{16^2 - (56 + 52 + 48)}\right\} = \frac{1}{2}(16 \pm 10)$
$= (13,\ \mathbf{3})$.

(ii). $\frac{1}{2}\left\{13 \pm \sqrt{13^2 - (60 + 28)}\right\} = \frac{1}{2}(13 \pm 9)$
$= (11,\ \mathbf{2})$.

(iii). $\frac{1}{2}\left\{11 \pm \sqrt{11^2 - (40)}\right\} = \frac{1}{2}(11 \pm 9) = (\mathbf{10},\ \mathbf{1}.)$

Thus the surd term obtained in the root are : $\sqrt{10}, \sqrt{3}, \sqrt{2}, \sqrt{1}$.

Contrary to the rule, if the surd terms in the root are obtained by subtracting first one integer equal to a surd number, then equal to two, and subsequently equal to three :

(i).). $\frac{1}{2}\left\{16 \pm \sqrt{16^2 - (60)}\right\} = \frac{1}{2}(16 \pm 14) = (15,\ \mathbf{1})$.

(ii). $\frac{1}{2}\left\{15 \pm \sqrt{15^2 - (56 + 48)}\right\} = \frac{1}{2}(15 \pm 11)$
$= (13,\ \mathbf{2})$.

(iii). $\frac{1}{2}\left\{13 \pm \sqrt{13^2 - (52 + 40 + 28)}\right\} = \frac{1}{2}(13 \pm 7)$
$= (\mathbf{10},\ \mathbf{3})$.

Thus surd terms obtained in the root are the same as above: $\sqrt{10}, \sqrt{3}, \sqrt{2}, \sqrt{1}$.

But the square of $(\sqrt{10} + \sqrt{3} + \sqrt{2} + \sqrt{1})$ is :

$$16 + \sqrt{120} + \sqrt{80} + \sqrt{40} + \sqrt{24} + \sqrt{12} + \sqrt{8}\ .$$

Hence, in such cases, Nārāyaṇa directs us to follow the alternative rule : (अत्र) सर्वत्र करणीनामासन्नमूलकरणेन मूलान्यानीय रूपेषु प्रक्षिप्य मूलं वाच्यम् ।

[**Note :** we got a wrong answer, since the computation did not conform to the rules laid down in verses 51-51$\frac{1}{2}$ (i.e., '**divisibility test**', and also '**quotients-identity test**')].

5.12: Rule for extracting square-root of a compound surd consisting of negative component surd terms :

सूत्रम् –

वर्गे क्षयात्मिक चेत्तामपि करणीं धनात्मिका कृत्वा ।
मूलं ग्राह्यं (च) तयोः क्षयात्मिकैका भवत्येव ॥५२॥

"If there be a negative surd in the square (expression), the traction of roots should be proceeded with supposing it as if positive; but of the two surds deduced, one chosen at pleasure by the intelligent mathematician, should be taken as negative."[20] ॥ **52** ॥

5.13 : Illustrative Examples :

Statement (*Nyāsa*): (i). Find the square root of : $8 - \sqrt{60}$.

There is a negative surd in the square (expression), so, according to the rule stated in the first half of verse-52, the extraction of roots should be proceeded with supposing it as if positive.

Hence, in accordance with the rule stated in verses 41-45, subtract from 64 , which is the square of the rational nmber 8, a number equal to the surd-number 60, the remainder is 4. Its square-root 2, added to and subtracted from, the rational number 8, makes 10 and 6. The moieties of which are 5 and 3, and therefore, the surds composing the roots are :$\sqrt{5}$, and $\sqrt{3}$.

Again according to the the rule stated in the second halfof the verse, among the two surds deduced, one should be taken as negative.

$\therefore \sqrt{(8 - \sqrt{60})} = (\sqrt{5} - \sqrt{3})$ *or* $(-\sqrt{5} + \sqrt{3})$.

[20] . Cf. BBi. ॥४२॥ 41॥ . *IJHS.* **28** (3), 1993. p. 259. (f.n. 27.)

(ii). Find the square root of : $\mathbf{10} + \sqrt{\mathbf{60}} + \sqrt{\mathbf{40}} - \sqrt{\mathbf{24}}$.

First, considering the negative surd in the square (expression), as if it is positive extract the square-root.

According to the rule for the extraction of square-root **:** From the square of the rational number 10, viz. 100, subtract numbers equal to two of the surds namely 40 and 24; the remainder is 36 ; and its square root 6, this subtracted from the natural number 10 and added to it, makes 4 and 16 ; the moieties of which are 2 and 8.

That is, $(1/2)\left[10 \mp \sqrt{\{10^2 - 24 + 40\}}\right] = (2\,,\ 8)$.

The first is a surd, $\sqrt{\mathbf{2}}$, in the root. Putting the second for a rational number, the same operation is again to be performed with the rest of the surds. From the square of this then treated as a rational number, 64, subtracting the number 60, the remainder is 4 ; and its root 2 ; which subtracted from the rational number and added to it, severally makes 6 and 10; the moieties where of are 3 and 5. They are surds in the root : $\sqrt{3}$, $\sqrt{5}$.

$$\text{That is, } (1/2)\left[8 \pm \sqrt{\{8^2 - 60\}}\right] = (\mathbf{5},\ \mathbf{3}).$$

$$\therefore \sqrt{10 + \sqrt{24} + \sqrt{60} + \sqrt{40}} = \left(\sqrt{2} + \sqrt{3} + \sqrt{5}\right).$$

Now assume one, are some among the component surds as negative.

$$\therefore \sqrt{10 + \sqrt{60} + \sqrt{40} - \sqrt{24}} = \left(-\sqrt{2} + \sqrt{3} + \sqrt{5}\right)$$
$$= \left(\sqrt{2} - \sqrt{3} - \sqrt{5}\right) .$$

If the component surd $\sqrt{3}$ is assumed as negative , then the computed square-root will be : $\left(\sqrt{2} - \sqrt{3} + \sqrt{5}\right)$.

But the square of this is : $10 - \sqrt{24} - \sqrt{60} + \sqrt{40}$, which is wrong.

Therefore, the positive and negative signs for the compoent surds deduced should be assumed in such a way that : the square of the computed root of the surd expression should be equal to the given surd expression.

इति करणीषड्विधम् ।

Thus ends the six operations on the surds

तदेवं षट्त्रिंशत्परिकर्माणि समाप्तानि ।

Like this, there ends the thirty-six operations.

Note : Algebraic operations dealt with in the work are addition, subtraction, multiplication, division, squaring and extracting the square-root for each of the following sections :

(i) Positive and negative numbers (*dhanarṇa*),

(ii) The zero (*śūnya*),

(iii) Single unknown (*avyakta*),

(iv) More unknowns characterized by colours (*varṇa*),

(v). surds (*karaṇī*).

Thus, operations in all works out to 30 only. Whereas the number of operations mentiond above are thirty-six (*ṣaṭtriṃśat -parikrmāṇi*).

अथ कुट्टकः

(PULVERISER)

INDETERMINATE EQUATIONS OF THE FIRST DEGREE

***Kuṭṭaka* :**

The subject of indeterminate analysis of the first degree that is, the solution of equation $by - ax = c$, for x, y in positive integers, where a, b, c are given integers is generally called *Kuṭṭa, Kuṭṭaka, Kuṭṭākāra, and Kuṭṭīkāra* etc. by the Hindus.

The equation $by - ax = c$, has been wrongly given the name "Diophantine equation", after the great Greek mathematician Diophantus (c.275 AD).

Diophantus was concerned with the determination of rational solutions of particular case of equations of the form $y^2 = ax^2 + bx + c$, but did not deal with linear indeterminate equations for integral solutions. It is now understood that: The name "Diophantine equation", given to this problem is a historical error[1].

Problems which led the Hindus to the investigation of such equations broadly fall into three varieties:

(i). In one variety, the problem is to find a number (N) which when divided by two given numbers (a, b) will leave two given remainders (R_1, R_2). Thus, symbolically,

$$N = ax + R_1 = by + R_2.$$

[1]. Srinivasiengar, C. N.:*The Hinstory of Ancient Indian Mathematics,* World Press, Calcutta, 1967. p.95.

or $$by - ax = (R_1 \sim R_2) = \pm C,$$

where $C = R_1 \sim R_2$.

In such type of problems (a, b) are called 'divisors', (*bhāgahāra, bhājaka and cheda* etc.) and (R_1, R_2) 'remainders' (*agra* and *śeṣa* etc.)

(ii). In the second variety, the problem is to find a number $\boldsymbol{x}$ such that its product with a given number $\boldsymbol{a}$ being increased or decreased by another given number $\boldsymbol{c}$ and then divided by third number $\boldsymbol{b}$ will leave no remainder. Thus, symbolically, the problem is to solve

$$\frac{(ax \pm c)}{b} = y, \text{ in integers.}$$

In such type of problems, generally,

b is called the 'divisor' (*bhājaka, hara etc.*),
a the 'dividend' (*bhājya*),
c the 'interpolator' (*kṣepa*),
x the 'multiplier' (unknown quantities to be obtained) (*guṇaka, guṇakāra etc.*),
y the 'quotient' (*labdhi, phala* etc.)

(iii).The third variety leads to the equations of the form, $by + ax = \pm c$.

Origin of the name and method of solution :

The Sanskrit words *Kuṭṭa, Kuṭṭaka, Kuṭṭākāra* and *Kuṭṭīkāra* are all derived from the root *kuṭṭ* "to crush", "to grind", "to pulverise" and hence etimologically they mean the act or process of "breaking", "grinding", "pulverizing" as well as an instrument for that, *i.e.* "grinder", "pulveriser".

Gaṇeśa in his *Buddhivilāsinī* (1545) [p.251] says :

(i) कुट्टककोनाम गुणकः । हिंसावाचकशब्दैर्गुणनाभ्युपगमात् ।

"Kuṭṭaka is a term for multiplier, for multiplication is admittedly called by words importing 'injuring', 'killing'."

(ii) कश्चिद्राशिर्येन गुणित उद्दिष्टक्षेपयुतोन उद्दिष्टहरभक्तः सन्निःशेषो भवेत्स गुणकः । कुट्टक इति पूर्वैर्व्यपदिश्यते । विशेष संज्ञेयम् ।

"A *certain* given number being multiplied by another (unknown quantity), added or subtracted by a given interpolator and then divided by a given divisor leaves no remainder ; that multiplier is the *Kuṭṭaka :* so it has been said by the ancients. This is a special technical term."

The same explanation as to the origin of the name *Kuṭṭaka* has been offered by Sūryadāsa (1538), Kṛṣṇa (c.1580), and Raṅganātha (1620).

The Hindu method of solving the equation

$$by - ax = \pm c$$

is essentially based on a process of deriving from it successively other similar equations in which the values of the coefficients $(a,\ b)$ become smaller and smaller. That is, formation of *valli,* and therefrom to get the solution. Thus the process is indeed the same as that of breaking a whole thing into smaller pieces. As opined by Datta and Singh, it is this that led the ancient Indian mathematicians to adopt the name *kuṭṭaka* for the operation.

Importance :

The earliest Hindu algebraist to give a treatment of *kuṭṭaka* (the indeterminate equation of first degree) is Āryabhaṭa-I (born. 476 C.E.). Beginning With Āryabhaṭa-I most of the Indian mathematicians dealt with the *kuṭṭaka,* or pulveriser method, to solve the above equations (**in positive integers**). These methods were also considered important in astronomy in finding the position of the planet and so on. *Kuṭṭaka* method is also used in solving some equations of several unknowns.

Though *Kuṭṭaka* belongs particularly to algebra, on account of its special importance, the treatment of this subject from his work *Bījagaṇita* has been repeated nearly word for word by Bhāskara II in his (treatise of arithmetic) *Līlāvatī.*

***Kuṭṭaka in Bījagaṇitāvatāṃsa* :**

Second section of *Bījagaṇitāvatāṃsa* deals with the indeterminate equation of the first degree (pulveriser), i.e., $ax \pm c = by$. In it first of all a rule (2.1) with regard to abridging the dividend, divisor and interpolator by a common factor, and necessary condition for integral solutions has been given inverse **53**. Then a rule (2.2) to make the dividend and the divisor prime to each other, so that the equation may be solvable (in integers) has been given in vese **54**. Next the method of obtaining the minimum solution and then from it, the general solution of the pulveriser, in different cases have been given (R.2.3 to 2.7) inverses **55-59.** Verse 60 (R.2.8) gives the solution

when either the divisor and the additive, or the dividend and the additive, or both, have a common factor (or different factor in the last case). A method of solution by working with the residue of division of either the additive or the dividend is dealt with in verses 61-61$\frac{1}{2}$ (R.2.9 and 2.10). A method of solution when the dividend is negative has been dealt with in verse 62 (R.2.11), whereas that when the additive is either an exact multiple of the divisor or cipher or the dividend is cpher in verse 63 (R.2.12). In verse 64 (R.2.13), the solution of an equation with the help of that of a constant pulveriser (i.e., an equation with unity as the additive or the subtractive) has been given. After that the solution of the problem when, in the rule of proportion, the argument , the fruit, and only a meager part of the fruit of the demand is given, (i.e. Rule for solution of Special astronomical problems) with the help of a pulveriser has been given in verses 65-67$\frac{1}{2}$ (R.2.14). Lastly 2 rules, an example, and a comment which are concerned with the pulveriser when its solution is fractional, and not found in *Gaṇitakaumudī,* are dealt with here in verses 68-69 (R.2.15).

[Note: Though the chapter on pulveriser in both the books *Gaṇitakaumudī* and *Bījagaṇitāvatāṃsa* is nearly the same, a set of four Sūtras followed by six examples dealing with the residual and conjunct pulverisers *sāgra-kuṭṭaka and saṁśliṣṭa- kuṭṭaka* occuring in *Gaṇitakaumudī* are not found here in *Bījagaṇitāvatāṃsa* (between verses 64 and 65).]

२. कुट्टकः

2.1: Abridging the dividend, divisor and interpolator by a common factor, and necessary condition for integral solutions

अथ कुट्टके सूत्रम् –

भाज्यो हारः क्षेपः केनाप्यपवर्त्य कुट्टकस्यार्थम् ।
येन विभाज्यच्छेदौ (छिन्नौ) क्षेपो न तेन खिलम् ॥५३॥

“For the investigation of the pulveriser, if possible, divide the dividend, the divisor and the interpolator by a common divisor. If the number by which the dividend and the divisor are divisible, does not divide the interpolator, the problem is incorrect. (i. e., it will not have integral solutions.)”[1] ॥ **53** ॥

2.2: Rule for Making the dividend and the divisor prime to each other for integral solutions :

हरभाज्ययोर्विहृतयोरन्योन्यं यो भवेद् ययोः शेषः ।स
तयोरपवर्तनकृत् (तौ) तेनैवापवर्तितौ तु दृढौ ॥५४॥

“Divide the dividend and the divisor, one by the other (and) the remainder of one by the remainder of the other. The (last) remainder is their common measure. Those two being divided by their common measure, become prime (*dṛḍha,* or *niccheda,* or *nirapavarta*) to each other.”[2] ॥ **54** ॥

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1 . (i).SiSe.. xiv-26.; (ii).L(ASS).Vs.242(ii).; (iii). BBi. R. 53(i). (iv).G.K.ix.19(ii).;(v).SiTVi.xiii.p.179f. Cf. HHM. II. pp. 92- 93.

2 . Cf. (i). MSi. Xviii- 1(i).; (ii). G.K. ix-20(ii).; (iii). BBi. R.53(ii).

2.3: Rule for construction of *valli* :

दृढभाज्यहरौ विभजेत् परस्परं यावदेकमवशेषम् ।
विन्यस्याधोऽधस्तात् फलानि तदधस्तथा क्षेपम् ॥५५॥
तदधः खमुपान्त्येनाहते निजोर्ध्वेऽन्तिमेन संयुक्ते ।
अन्त्यं जह्यादेवं यावद्राशिद्वयं भवति ॥५६॥

"Divide mutually the dividend and the divisor, made prime to each other, until the remainder is 1. Set down the quotients (of mutual division), one below the other (successively), benaeath them the interpolator and the cipher below that. Multiply by the penultimate, the number just above it (and) add the ultimate (to the product). Then, reject the ultimate. (Do so repeatedly) until only two numbers are left."[3] ॥ **55-56** ॥

2.4: Rule of obtaining minimum solution for the even number of quotients, and for the odd number of quotients obtained in the mutual division of the dividend and the divisor (when the interpolator is positive) :

हरभाज्याभ्यां तष्टावधरोध्वौ ते क्रमेण गुणलब्धी ।
यदि लब्धयः समाः स्युस्तदा गुणाप्ती यथागते भवतः ॥५७॥
विषमाश्चेत् ते शोध्ये गुणलब्धी स्वस्वतक्षणाच्छेषे ।

"Divide the lower one by the reduced divisor, like the upper one by the reduced dividend. The remainders are the multiplier (and) the quotient, in order. If the number (of mutual division) be even (the remainders) so obtained are the multiplier and the quotient. If that is

[3] . Cf. (i). BBi. R. 55-56(i). (ii). G.K. ix.21-22. ;

odd, subtract the multiplier and the quotient, from their respective abraders."[4] ॥ **57-58(i)** ॥

2.5: Rule of obtaining minimum solution when the interpolator is negative :

योगभवे गुणलब्धी निजतक्षणतो विशोधिते क्षयजे ॥५८॥

"(If the interpolator is) positive, the remainders are the multiplier and the quotient but (if the interpolator is) negative, (these) subtracted from their respective abraders, (are so)."[5] ॥ **58(ii)** ॥

2.6: Rule for General solution :

इष्टघ्नतक्षणयुते बहुधा भवतो गुणाप्ती ते ।

"These added to (the product of) any desired number multiplied by the abraders become manifold."[6] ॥ **59(i)** ॥

2.7: Rule for abrading while computing multiplier and quotient :

सर्वत्र कुट्टकविधौ कार्यं समतक्षणं सुधिया ॥५९॥

"In all methods, the learned should take the same (quotient) in (both the) abrading."[7] ॥ **59(ii)** ॥

That is to say,

(i). If $x = \alpha,\ \ y = \beta$ be the minimum solution of $ax + c = by,$ obtained by writing , $\alpha' \equiv \alpha \bmod b,$ $\beta' = \beta \bmod a$ is correct only when

[4] . Cf. (i). BBi. R. 56(ii). (ii). G.K. ix.23-24(i). ;
[5] . Cf. (i). L(ASS). Vs. 250. ; (ii). G.K. ix. 24(ii) ; (iii). BBi.R.59(i).
[6] . Cf. (i). L(ASS).Vs.-256. (ii).G.K. ix-25(i). (iii).BBi.R.64.
[7] . (i). L(ASS). Vs. 252(i). (ii). G.K. ix. 25(ii). (iii). BBi.R.60.
Cf. HHM. II. p. 112.

$\frac{\alpha'-\alpha}{b} = \frac{\beta'-\beta}{a} =$ the same integer.

[Def. of mod : If $a = qm + r$; where $0 \le r < m$, then $a \equiv r \ mod \ m$.]

Kuttaka method (Explanation) :

Consider the equation,

$$ax \pm c = by . \quad \ldots\ldots(1).$$

Here, a is the 'dividend' (*bhājya*), b is the 'divisor' (*bhājaka, hara etc.*), c the 'interpolator' (*kṣepa*), x the 'multiplier' (*guṇaka guṇakāra etc.*) (i.e.,unknown quantities to be obtained) y the 'quotient' (*labdhi, phala* etc.).

In order that the equation (1) may be solvable for integral values, the numbers $a,\ b,\ c$ must be made prime to each other (i.e., *dṛḍha, niccheda, nirapavrta*).

Step.1: Making a, b, c relatively prime :

If the dividend, the divisor and the interpolator have a common factor, divide them by their common factor.

If any number, divides both the dividend and the divisor and does not divide the interpolator, then the problem is incorrect. However the condition is necessary only in case integral values of x and y are sought.

Step.2 : Mutual division of the dividend and the divisor :

The reduced dividend (*dṛḍha bhājya*) is divided by reduced divisor (*dṛḍha bhājaka*) mutually untill 1 is obtained as a remainder. Thus a series of quotients are obtained.

Step.3 : Construction of *valli* :

Taking the quotients obtained in the mutual division in their order, next the interpolator, and the cipher as the last number are kept one below the other ; thus a coloumn of numbers is formed. Now, the number just above the penultimate (*upāntyūrdhva*) is multiplied by the penultimate (*upantya*) and the ultimate (*antya*) is added to the product. By this sum replace the number just above the penultimate (*i.e.,the upāntyūrdhva*); then the ultimate is rejected. Thus another column of numbers is obtained. The process is repeated until only two numbers are left.

Step.4 : To find 'quotient' y (*labdhi, phala*), and 'multiplier' x (*guṇaka, or guṇakāra etc.*),:

(i). The upper number in the last column of the *valli* being abraded (or divided) by the reduced dividend, the remainder is the quotient y ; and the lower one being divided by the reduced divisor, the remainder is the multiplier x ; if the number of quotients of mutual division is even.

(ii). If the number of quotients of mutual division is odd, the quotient and multiplier so obtained must be subtracted from their respective abraders (i.e., reduced dividend and reduced divisor) to get the true quotient and multiplier.

[**Note:** In abrading the (calculated values of) the multiplier (*guṇa*) and the quotient (*labdhi*) by their *takṣaṇa* (i. e., divisor and the dividend respectively) one should take out the same multiple of them.]

(iii) If the interpolator is negative, the quotient and the multiplier are obtained by considering it as positive. The quotient and the multiplier, so obtained, are subtracted

from the dividend and divisor, the remainders are the quotient and the multiplier, respectively, for the negative interpolator.

In any one of the above 3 cases, any integral multiple of the dividend and the same integral multiplier of the divisor, added to the quotient and the multiplier are also the quotient and the multiplier, respectively, for that case.

Rationale [8]: Consider the equation,

$$ax + c = by \quad \ldots\ldots\ldots\ldots\ldots (1)$$

a being the dividend, c the additive and b the divisor. Also, suppose that a and b are prime to each other and $b < a$. [If $b > a$ we will have, $q = 0$ and $r_1 = a$; for q and r_1, see below]

We perform the usual Euclidean algorithm on a/b .

That is, mutual division of $a,\ b$

Let

b) a (q
<u>bq</u>
r_1) b (q_1
$r_1 q_1$
r_2) r_1 (q_2
$r_2 q_2$
r_3
. … …. …..
r_{m-1}) r_{m-2} (q_{m-1}
$r_{m-1} q_{m-1}$
r_m) r_{m-1} (q_m
$r_m q_m$
r_{m+1}

8 . HHM.II. pp. 96-99.

Then, we get

$$a = bq + r_1,$$
$$b = r_1 q_1 + r_2,$$
$$r_1 = r_2 q_2 + r_3,$$
$$r_2 = r_3 q_3 + r_4,$$
$$\ldots\ldots\ldots\ldots$$
$$r_{m-2} = r_{m-1} q_{m-1} + r_m,$$
$$r_{m-1} = r_m q_m + r_{m+1}$$

Now substituting the value of a in (1), we get,

$$by = (bq + r_1)x + c.$$

Therefore $y = qx + y_1,$ where $y_1 = \frac{r_1 x + c}{b}$.

In other words,

since $a = bq + r_1$, on putting $y = qx + y_1$,

the equation (1) reduces to $by_1 = r_1 x + c$. ... (1.1)

Again , since $b = r_1 q_1 + r_2$,

putting similarly $x = q_1 y_1 + x_1$

the equation (1.1) reduces to

$$r_1 x_1 = r_2 y_1 - c \quad \ldots\ldots\ldots (1.2)$$

and so on. Writing down the successive values and reduced equations in columns, we have

(1)	$y = qx + y_1,$	$by_1 = r_1 x + c,$	1.1
(2)	$x = q_1 y + x_1,$	$r_1 x_1 = r_2 y_1 - c,$	1.2
(3)	$y_1 = q_2 x_1 + y_2,$	$r_2 y_2 = r_3 x_1 + c$	1.3
(4)	$x_1 = q_3 y_2 + x_2,$	$r_3 x_2 = r_4 y_2 - c,$	1.4
(5)	$y_2 = q_4 x_2 + y_3,$	$r_4 y_3 = r_5 x_2 + c,$	1.5
(6)	$x_2 = q_5 y_3 + x_3,$	$r_5 x_3 = r_6 y_3 - c,$	1.6
	………………	…………………	

(2n-1)	$y_{2n-1} = q_{2n-2}x_{n-1} + y_n$	$r_{2n-2}y_n = r_{2n-1}x_{n-1} + c$	1. (2n-1)
(2n)	$x_{n-1} = q_{2n-1}y_n + x_n,$	$r_{2n-1}x_n = r_{2n}y_n - c$	1. (2n)
(2n+1)	$y_{2n+1} = q_{2n}x_n + y_{n+1}$	$r_{2n}y_{n+1} = r_{2n+1}x_n + c$	1.(2n+1)

Now, suppose the mutual division is continued until the remainder is unity.

Case (i). If the number of quotients be even, we then have

$$r_{2n} = 1, \quad r_{2n+1} = 0, \quad q_{2n} = r_{2n-1}.$$

The equations 1.(2n) and 1.(2n+1), therefore, become

$$y_n = q_{2n}x_n + c$$

and $$y_{n+1} = c$$

respectively. Giving an arbitrary integral value (t) to x_n, we get an integral value of y_n. From that we can find the value of x_{n-1} with the help of the equation (2n). Proceeding backwards step by step we ultimately find the values of x and y in positive integers.

Case(ii). If the number of quotients be odd, we will have,

$$r_{2n-1} = 1, \; r_{2n} = 0, \text{ and } q_{2n-1} = r_{2n-2}.$$

The equations (2n+1) and 1.(2n+1) will then be absent and the equations (2n) and and 1.(2n) will be reduced respectively to

$$x_{n-1} = q_{2n-1}.y_n - c\,;$$

and $$x_n = -c\,.$$

Giving an arbitrary integral value (t') to y_n we get an integral value of x_{n-1}. Then proceeding back wards as before we can calculate the values of x and y in positive integers.

General solution :

Any integral multiple of the reduced dividend added to the quotient, and the same integral multiple of the reduced divisor, added to the multiplier respectively, are also a pair of quotient and multiplier of the given *Kuṭṭaka.*

Āryabhaṭa I has given a rule for solving $\frac{ax+c}{b} = y$ by stopping at any finite desired number of steps in mutual division of the dividend and the divisor[9]. By considering the equation, $\frac{ax-c}{b} = y$, where y is positive integer, Bhāskara I supplements the form of Āryabhaṭa I, in which the interpolator is always made positive by necessary transposition[10]. The same rule which holds under the same conditions has also been given by Srīpati[11]. Bhāskara I has also given[12] a rule for solving $\frac{ax\pm c}{b} = y,$

where $a = mb \pm a'$ by proceeding at once with the solution of the equation $a'x \pm c = by.$

Brahmagupta has given a rule[13] for solving Āryabhaṭa's problem. Mahāvīra has given a rule [14] with a view to find the solution of $\frac{ax\pm c}{b} = y,$ in positive integers. The same method has been redescribed by him in a

9 Ā-ii-32-33. Cf. HHM, II. pp.93-99

10 MBh.i.42-44. ; Cf. HHM,II. pp. 99-100.

11 SiSe.xiv.22-25. ; Cf.HHM. II. p. 110.

12 MBh. i-47. Cf. HHM. II. p.101.

13 BrSpSi.xviii.3-5, and 13. Cf. HHM. II. pp.101-103.

14 GSS.vi.115½ . Cf. HHM. II. p.103

slightly different form.[15] Here, in this case, he continues the mutual division until cipher is obtained as a remainder and further takes the optional multiplier to be cipher.

उदाहरणम् –

राशिस्त्रिसप्ततियुतेन शतद्वयेन
निघ्नो नवोनितशतेन युतश्च कोऽपि ।
भागं प्रयच्छित विशुद्धमगाब्धिनेत्रै–
र्भक्तः सखे क्थय तं गुणकं फलं मे ॥२६॥

Ex.26: "A number is multiplied by 273 and (the product) is increased by 91. (The sum is exactly divisible by 247. Oh friend, tell me quickly that number and also the quotient."[16] ॥ **26** ॥

Statement (*Nyāsa*):

Here, the dividend =273, the additive =91 and the divisor=247.

∴ the equation to be solved is $273x + 91 = 247y$.

Solution :

The dividend, the divisor and the interpolator have a common factor 13.

```
247 ) 273 ( 1
      247
       26 ) 247 ( 9
            234
             13 ) 26 ( 2
                  26
                  00
```

Reducing them by 13, the dividend =21, the additive =7 and the divisor=19 , so the equation formed will be :

[15] GSS.vi.136½ . Cf. HHM. II. pp.103-104

[16] . GK. IX, Ex. 21.

$$21x + 7 = 19y$$

The mutual division of the dividend and the divisor willbe as follows :

```
19 ) 21 ( 1
     19
      2 ) 19 ( 9
          18
           1
```

so that the quotients obtained are 1 and 9. Also the additive is 7, putting cipherat the end, the coloum formed will be the first coloumn in the following table (i.e., *valli*).

Addition of the lowest i.e., 0 to the product of the penultimate i.e., 7 and the number above it, i.e., 9, gives $9 \times 7 + 0 = 63$. Now the lowest 0 is rejected and the number 9 in the first column is replaced by 63. Thus the second column is obtained. Again, by the same process, we get $1 \times 63 + 7 = 70$ which replaces 1 and the 7is rejected. Thus obtained is the third coloumn.

1	1	$1 \times 63 + 7 = 70$
9	$9 \times 7 + 0 = 63$	63
7	7	-
0	-	-

Since only two numbers are left in the last column, construction of valli is complete.

Now the upper number in the last column is abraded by the dividend, 21, and the lower one by the divisor, 19. The quotient of abrading the upper and the lower numbers in the last column of the *valli* being the same 3, in each case, and also the number of quotients of mutual division (=2) is even, the residues obtained are the

minimum values of the quotient, y , and the multiplier x , respectively.

Any integral number, multiplied by the divisor , 19, the product added to (subtracted from) the multiplier will also be a multiplier. Similarly, the same integral number multipliedby the dividend, 2, the product added to (or subtracted from) the qoutient, 7, will also be the corresponding quotient. Thus the minimum solution of (1) is $x = 6,\ y = 7$ and its general solution is

$x = 6 \pm 19t,\ y = 7 \pm 21t,$ where t is any integer.

Verification:

t	x	y	273x+91	247y	remark
1	6+19=25	7+21=28	273× 25+ 91= 6916	247× 28= 6916	correct
2	6+38=44	7+42=49	273× 44+ 91= 12103	247× 49= 12103	correct
3	6+57=63	7+63=70	273× 63+ 91= 17290	247× 70= 17290	correct

It has been shown that the process of solving a problem by the method of the pulveriser (i. e., *kuṭṭaka*), can some times be abbreviated to a great extent.[17]

[17] . HHM. II. pp. 111-113.

2.8: Rule for Reducing divisor (*bhājaka*) and additive (*kṣepa*), or dividend (*bhājya*), and additive (*kṣepa*), or both by their common divisor:

सूत्रम् –

हारक्षेपकयावा प्रक्षेपकभाज्ययोस्तदुभयोर्वा ।
अपवर्तित्ज्योर्गुणको लब्धिश्च स्वपवर्तहते ॥६०॥

"The divisor and the additive or 'the dividend and the additive' of 'both' may be reduced by a common (or different in the last case) divisor. The multiplier and the quotient (obtained), multiplied by their respective divisors, are the (true ones)."[18] ॥ **60** ॥

Let the equation be $ax \pm c = by$. ……. (1).

The numerical work can be reduced if the numbers $(a,\ c)$ or $(b,\ c\,)$ have a common factor.

(i) In the equation $ax + c = by$, suppose $a = ka'$, $c = kc'$ where k is an integer. Then the equation reduces to $a'x + c' = by'$, where $y' = y/k$.

If $(x,\ y')$ are solutions of the reduced equation $a'x + c' = by'$, then $(x,\ ky')$ are solutions of the original equation $ax + c = by$.

(ii) In the equation $ax + c = by$, suppose $b = kb'$, $c = kc'$ where k is an integer. Then the equation reduces to $ax' + c' = b'y$, where $x' = x/k$.

If $(x',\ y)$ are solutions of the reduced equation : $ax' + c' = b'y$, then $(kx',\ y)$ are solutions of the original equation $ax + c = by$.

[18] . Cf. (i). L(ASS). Vs.248. ; (ii).G.K. ix-26. ; (iii). BBi. R. ॥५३॥ 58 ॥

(iii) It may be possible to combine cases (i) and (ii).

Let $a = ka'$, $c = kc'$

$b = lb''$, $c' = lc''$,

Then the equation $ax + c = by$ becomes

$$ka'x + klc'' = lb''y$$

or $a'X + c'' = b''Y$, where $X = \frac{x}{l}$, $Y = \frac{y}{k}$.

If (X, Y) are solutions of the last equation, then (lX, KY) are solutions of the given equation $ax + c = by$.

उदाहरणम् –

येनाभिहताऽशीतिः समन्विता त्रिंशता च वियुता वा ।
त्रिगुणत्रयोदशाप्ता शुध्यति तं कथय पृथगाप्तिम् ॥२७॥

Ex.27: **"**Anumber is multiplied by 80, (and then) 30 is either added to or subtracted (from the product). The sum (or the difference) becomes exactly divisible by 39. Tell (that number and) the quotient, separately."[19] ॥ **27** ॥

Statement (*Nyāsa*) : Here, the dividend = 80 and the interpolator = 30 and the divisor = 39, so the equation will be : $80x \pm 30 = 39y$ ….. … … … … (1)

Solution :

Method – I : Whithout reducing the dividend, the divisor, and the interpolator (by their common factor):

Case –(i): With positive interpolator, i.e., $80x + 30 = 39$. Mutual division of the dividend and the divisor will give

[19] . GK. IX, Ex. 22.


```
3 9 ) 8 0 ( 2
      7 8
        2 ) 3 9 ( 1
            3 8
              1
```

the quotients 2 and 1 in succession. Here the interpolator is 30, constructing the valli as stated verses 55-56 it will be as follows :

2	2	2× 570+ 30= 1170
19	19× 30+ 0= 570	570
30	30	-
0	-	-

The number of quotients of mutual division being 2, an even number , and also,

570 = 546 + 24 or 570=39 × 14 + 24 ;

1170= 1120 + 50 or 1170 = 80 × 14 + 50.

By the rule stated in verses 57 -58(i), the minimum solution of (1) is $x = 24\ and\ y = 50$.

Now in accordance with the rule stated in verse 59(i).the general solution of (i) is

$x = 24 + 39t\ and\ y = 50 + 80\ t$, t being any integer.

Case (ii) : $80x - 30 = 39$.

In accordance with rule stated in verse 58(ii) , subtracting 24 and 50 from their corresponding abraders we get the minimum solution as x = 39 - 24 =15, and y = 80-50 =30. Therefore, the general solution in case of (ii) with negative interpolator is :

$x = 154 + 39t\ and\ y = 30 + 80\ t$, t being any integer.

Method–II: Reducing the dividend, and the interpolator by their common factor :

Reducing the dividend, and the interpolator by their common factor 10 the given eqution (1) reduces to

$$8x \pm 3 = 39y$$

Case-(i): $\mathbf{8x + 3 = 39y}$ .... $\mathbf{(2)}$

By mutual division of the reduced dividend and the divisor

```
39 ) 8 ( 0
     0
     8 ) 39 ( 4
         32
          7 ) 8 ( 1
              7
              1
```

we get the quotients 0, 4, and 1 in succession. Here the reduced interpolator being 3, constructing the valli as stated verses 55-56 it will be as follows :

0	0	0	0× 15 + 3 = 3
4	4	4× 3+ 3= 15	15
1	1× 3+ 0= 3	3	-
3	3	-	-
0	-	-	-

$$15 = 39 \times 0 + 15; \quad 3 = 8 \times 0 + 3$$

Since the number of quotients of mutual division (=3)is odd, subtracting the 3 and 15 from their corrresponding abraders, we get 39 - 15 = 24 ; 8 – 3 = 5. So, x = 24 ; y = 5 are the minimum solution of the reduced equation (2).

So in accordance with the rule stated in verse 60, the minimum solution of $80x + 30 = 39$, is $x = 24$ and $y = 5 \times 10$ (the common factor of the dividend and the interplator) =50.

Now in according to the rule stated in verse 59(i). the general solution of (i) is
$x = 24 + 39t$ *and* $y = 50 + 80\,t$, t being any integer.

Case-(ii): The general solution in case of (ii) [with negative interpolator], i.e. $80x - 30 = 39$ is :
$x = 154 + 39t$ *and* $y = 30 + 80\,t$, t being any integer.

Method–III: Reducing the divisor , and the interpolator by their common factor :

Reducing the divisor, and the interpolator by their common factor 3 the given eqution (1) reduces to

$$80x \pm 10 = 13y \quad \ldots\ldots\ldots (3).$$

By mutual division of the dividend and the reduced divisor:

```
13 ) 80 ( 6
     78
      2 ) 13 ( 6
          12
           1
```

we get the quotients 6, and 6 in order. Here the reduced interpolator being 10, constructing the valli as stated verses 55-56 it will be as follows :

6	6	6× 60+ 10= 370
6	6 × 10 + 0 = 60	60
10	10	-
0	-	-

The number of quotients of mutual division being 2, an even number , and also,

$$370 = 80 \times 4 + 50 \quad OR \quad 370 \equiv 50 \ mod\ 80;$$
$$60 = 13 \times 4 + 8 \quad OR \quad 60 \equiv 8 \ mod\ 13\ .$$

By the rule stated in verses 57 -58(i), the minimum solution of $80x + 10 = 13y$ is $x = 8\ and\ y = 50$.

Now, , in accordance with the rule stated in verse 60, the minimum solution of $80x + 30 = 39$is

$x = 8 \times 3$ (the common factor of the divisor and the interplator) = 24 and $y = 50$.

Now in accordance with the rule stated in verse 59(i). the general solution of (i) is

$x = 24 + 39t\ and\ y = 50 + 80\ t,$ t being any integer.

Case-(ii): The general solution in case of (ii) [i.e., with negative interpolator], is :

$x = 15 + 39t\ and\ y = 30 + 80\ t,$ t being any integer.

Method–IV: First Reducing the divisor , and the interpolator by their common factor. and then again reducing the reduced interpolator and the dividend by their common factor:

First Reducing the divisor , and the interpolator by their common factor 3, and then again reducing the reduced interpolator and the dividend by their common factor 10 , we have, the dividend = 8, the divisor = 13: and the interpolator = 1, and so the reduced equation will be

$$8x \pm 1 = 13y \quad \ldots\ldots\ldots.(4)$$

Mutual division of the dividend and the divisor will give the quotients 0,1,1,1, and 1 in succession.


```
1 3 ) 8 ( 0
      0
      8 ) 1 3 ( 1
            8
            5 ) 8 ( 1
                5
                3 ) 5 ( 1
                    2 ) 3  ( 1
                        2
                        1
```

0	0	0	0	0	0× 5+ 3= 3
1	1	1	1	1× 3+ 2= 5	5
1	1	1	1× 2+ 1= 3	-	-
1	1	1× 1+ 1= 2	2	-	-
1	1× 1+ 0= 1	1	-	-	-
1	1	-	-	-	-
0	-	-	-	-	-

Since the number of quotients of mutual division (=5), is odd, subtracting the 3 and 5 from the corresponding abraders, we get , y = 8 – 3 = 5 and x = 13 – 5 = 8 as the minimum solution of $8x + 1 = 13y$ (4).

Multiplying the value of the x by the common measure 3 of interpolator and of the divisor, and the value of the quotient y by the common measure 10 of the dividend and the reduced interpolator separately an in order, $x = 8 \times 3 = 24$ and $y = 5 \times 10 = 50$.

Now in accordance with the rule stated in verse 59(i). the general solution of (i) is

$x = 24 + 39t\ and\ y = 50 + 80\ t,$ t being any integer.

Case-(ii): The general solution in case of (ii) [i.e., with negative interpolator], is :
$x = 15 + 39t$ *and* $y = 30 + 80t$, t being any integer.

Answers :
Solution of $80x + 30 = 39$:

t	1	2	....
x = 24 + 39t	63	102	...
y = 50 +80t	130	210	....

Solution of $80x - 30 = 39$:

t	1	2	....
x = 15 + 39t	54	93	...
y = 30 +80t	110	190	....

उदाहरणम् –

को राशिः पञ्चभिः क्षुण्णः सप्तत्रिंशत्समन्वितः ।
वर्जितो वा त्रिभिर्भक्तो निरग्रः स्याद् वदाशु तम् ॥२८॥

Ex.28 : "Anumber is multiplied by 5, (and then) 37 is added to or subtracted from the product. (The result) is exactly divisible by 3. Tell that number (and the quotient,) quickly."[20] ॥ **28** ॥

Statement (*Nyāsa*) : Here, the dividend = 5 and the interpolator = 37 and the divisor = 3, so the equation will be

$5x \pm 37 = 3y$ .... (1)

Solution : Case- 1: $5x + 37 = 3y$
Mutual division of the dividend and the divisor will give

[20] . GK. IX, Ex. 23.

3) 5 (1
3
2) 3 (1
2
1

1	1	1× 37+ 37= 74
1	1× 37+ 1= 37	37
37	37	-
0	-	-

Now, abrading the upper number 74 ,by the dividend 5, will give the quotient 14 and abrading of the lower number 37, by the divisor 3, will give the quotient 12.

Here the quotients are different. but we must take the residues with the same quotient . Therefore take 12 as the quotient in both the cases :

37=12× 3+1; and 74 =12× 5+14 , therefore the minimum solution is ,

The multiplier = x = 1, and the quotient y = 14.

If the quotient of abrading is taken as 14,
then $37 = 14 \times 3 - 5$, and $74 = 14 \times 5 + 4$,
∴ The multiplier = $x = -5$, and the quotient $y = 4$.

Now in accordance with the rule stated in verse 59(i). the general solution of $5x + 37 = 3y$ is
$x = 1 + 3t$; $y = 14 + 5t$ or $x = -5 + 3t$; $y = 4 + 5t$.

Case- 2: $5x - 37 = 3y$

The general solution in case-2 [i.e., with negative interpolator], is :
$x = (3 - 1) + 3t$ *and* $y = (5 - 14) + 5t$, or
$x = (3 - (-5)) + 3t$ *and* $y = (5 - 4) + 5t$, or
i.e. $x = 8 + 3t$; $y = 1 + 5t$; t being any integer.

Here, Nārāyaṇa comments,

समतक्षणमित्युपचारो यथेष्टघ्नतक्षणयुते बहुधा गुणाप्ती भवतस्थथेष्टघ्नतक्षणवियुते (राशिद्वये) बहुधा गुणाप्ती भवतः ।

"The practice of (taking) the same (quotient) for the abrader means any optional (integral) multiple of the abrader added to (the multiplier and the quotient), the multiplier and the quotient become many fold and (also) any optional (integral) multiple of the abrader subtracted (from the two number), the multiplier and the quotient become many fold."

2.9: Alternative rule to find actual multiplier and quotient when the interpolator is abraded by the divisor :

सूत्रम् –

हरतष्टे धनक्षेपे ल्ब्धिस्तक्षणफलेन संयुक्ता ।
क्षयगे क्षेपे तक्षणफलोनिते जायते लब्धिः ॥६१॥

"Abrade the interpolator by the divisor (and work with the residue in place of the interpolator). The multiplier and quotient may be found as before; the quotient (obtained), however, must be increased by the abrading quotient in case the interpolator is positive, but, if it is negative, the abrading quotient must be subtracted. (But there will be no change in the multiplier)."[21] ॥ **61** ॥

That is,

In the equation $ax \pm c = by$, suppose that $c = bq + s$; if $x = m, y = n$ is the minimum solution of $ax \pm s = by$, then $x = m$, $y = n \pm q$ is the minimum solution of $ax \pm c = by$.

[21]. Cf. (i). L(ASS).Vs.252 (ii-iii).(ii). G.K. IX.-27. (iii). BBi.R.॥५६॥ 61॥ ; HHM. II. p. 112.

2.10: Alternative rule to find actual multiplier and quotient when the dividend is abraded by the divisor:[22]

हरतष्टभाज्यराशौ तष्टफलघ्नगुणसंयुता लब्धिः ॥ $61\frac{1}{2}$ ॥

"Abrade the dividend by the divisor (and work with the residue in place of the dividend). The quotient is the sum (of the quotient obtaimed from working with the residue and the product of) the multiplier and the result (i.e., the quotient of abrading)" ॥ **$61\frac{1}{2}$** ॥

In the equation : $ax \pm c = by$, suppose that $a = bp + r$ and let $x = m$, $y = n$ be the solution of $rx \pm c = by$ then according to the rule,

$x = m, y = n \pm mp$ wil be a solution of $ax \pm c = by$.

For the special case when both dividend and interpolators are greater than the divisor :

The multiplier may be found as before after abrading both the dividend and the interpolator by the divisor ; from (this multiplier) the quotient may be found by multiplying (it) by the dividend, adding the (interpolator) and then dividing (the sum by divisor). (That is, by substituting the value of the multiplier in the original equation.)[23]

That is, in the equation $ax \pm c = by$, suppose that $a = bp + r$ and $c = bq + s$; and also if $x = m, y = n$ is the minimum solution of $rx \pm s = by$,

then $x = m$, $y = n + mp \pm q$ is the minimum solution of $ax \pm c = by$.

[22] . G.K. IX.-28(i).
[23] . HHM. II. p.113.

उदाहरणम् –

को राशिः खाभ्रदिङ् निघ्नो दिगश्विनयनैर्युतः ।
हीनो वाऽग्नीन्द्रसम्भक्तः शुध्यित ब्रूहि तं पृथक् ॥२९॥

Ex. 29: "A number is multiplied by 1000. (When) 2210 is either added to or subtracted from the product. (The sum or difference) becomes exactly divisible by 143. Tell them, separately."[24]॥ **29** ॥

Statement (*Nyāsa*) : Here, the dividend = 1000 and the interpolator = 2210 and the divisor =143, so the equation will be $1000x \pm 2210 = 143y$ ….. … … … … (1)

Solution : Case i. $\mathbf{1000x + 2210 = 143y}$
Method – I : Whithout reducing the dividend, the divisor, and the interpolator (by their common factor):

```
1 4 3 ) 1 0 0 0 ( 6
          8 5 8
          1 4 2 ) 1 4 3 (1
                  1 4 2
                      1
```

6	6	6× 2210+ 2210= 15470
1	1× 2210+ 0= 2210	2210
2210	2210	-
0	-	-

The number of quotients of mutual division being 2, an even number , and also,

$$2210 = 143 \times 15 + 65 ; \quad 15470 = 1000 \times 15 + 470$$

By the rule stated in verses 57 -58(i), the minimum solution of (1) is $x = 65$ and $y = 470$.

[24] . GK. IX, Ex. 23

Now in accordance with the rule stated in verse 59(i).the general solution of (i) is $x = 65 + 143t$ and $y = 4700 + 1000\,t$, t being any integer.

Case (ii) : $\mathbf{1000x - 2210 = 143y}$ ……..(**2**)

In accordance with rule stated in verse 58(ii) , subtracting 65 and 470 from their corresponding abraders we get the minimum solution as x = 143-65=78, and y = 1000-470=530. Therefore, the general solution in case of (ii) with negative interpolator the general solution is :

$$x = 78 + 143t \text{ and } y = 530 + 1000t,$$

t being any integer.

Method II : Alternatively when the interpolator is abraded by the divisor :

Division of 2210 by 143 gives, the quotient 15 and remainder 65. The abridged equation is $1000x + 65 = 143y$

```
1 4 3 ) 1 0 0 0 ( 6
          8 5 8
          1 4 2 ) 1 4 3 ( 1
                  1 4 2
                      1
```

6	6	6× 65 + 65 = 455
1	1× 65 + 0 = 65	65
65	65	-
0	-	-

$$65 = 0 \times 143 + 65; \therefore m = 65;$$

$$455 = 0 \times 1000 + 455 \therefore n = 455$$

Thus the solution of equation (1), in accordance with the rule stated in verse 61 is $(x,\ y) = (65,\ 455 + 15 = 470)$.

Method III : When the dividend is abraded by the divisor:

$$a = b \times p + r, \quad \text{i.e.} \quad 1000 = 143 \times 6 + 142.$$

The abridged equation is $142x + 2210 = 143y$

```
1 4 3 ) 1 4 2 ( 0
            0
        1 4 2 ) 1 4 3 ( 1
                1 4 2
                    1
```

0	0	0 × 2210+ 2210= 2210
1	1× 2210+ 0= 2210	2210
2210	2210	-
0	-	-

$2210 = 143 \times 15 + 65;\ \therefore m = 65$;
$2210 = 142 \times 15 + 80;\ \therefore n = 80$

Then according to the rule stated in verse $61\frac{1}{2}$,
$x = m = 65,\ \ y = n \pm mp = 80 + 65 \times 6 = 470$;

will be a solution of $1000x + 2210 = 143y$.

Method IV: When the dividend and the interpolator both are abraded by the divisor:

$1000x + 2210 = 143y$ .... ①

Here, $a = b \times p + r$,

$1000 = 143 \times 6 + 142$; and

$c = b\ q + s$
$2210 = 143 \times 15 + 65$

The abridged equation is : $142x + 65 = 143y$


```
1 4 3 ) 1 4 2 ( 0
            0
        1 4 2 ) 1 4 3 ( 1
                1 4 2
                    1
```

0	0	0 × 65 + 65= 65
1	1× 65 + 0 = 65	65
65	65	-
0	-	-

$65 = 0 \times 143 + 65, \therefore m = 65$; $65 = 0 \times 142 + 65, \therefore n = 65$.

$(x,\ y) = (65,\ 65)$ is the solution of $142x + 65 = 143y$

Then according to the rule stated in verse 60 and $61\frac{1}{2}$,

$x = m = 65, y = n \pm mp = 65 + 65 \times 6 + 15 = 470.$

Case ii. $\mathbf{1000x - 2210 = 143y}$

1. When the dividend is abraded by the divisor:

$$a = b \times p + r, \quad \text{i.e.} \quad 1000 = 143 \times 6 + 142.$$

The abridged equation is $142x - 2210 = 143y$

The minimum solution of $142x + 2210 = 143y$ is $(x,\ y) = (65,\ 80)$. In accordance with rule stated in verse 58(ii) , subtracting 65 and 80 from their corresponding abraders we get the minimum solution as x = 143-65=78, and y = 142 – 80 =62.

Then according to the rule stated in verse $61\frac{1}{2}$,
the solution of (2) will be $x = 78, y = 62 + 6 \times 78 = 530$.
Therefore, the general solution in case of (ii) with negative interpolator the general solution is :
$x = 78 + 143t\ and\ y = 530 + 1000\ t$,
t being any integer.

2. When the interpolator is abraded by the divisor:

Division of 2210 by 143 gives, the quotient 15 and remainder 65.
The abridged equation is $1000x - 65 = 143y$.
The minimum solution of $1000x + 65 = 143y$ is
$(x, y) = (65, 455)$.

In accordance with rule stated in verse 58(ii) , subtracting 65 and 455 from their corresponding abraders we get the minimum solution of

$1000x - 65 = 143y$

as x = 143-65=78, and y = 1000 – 455 =545.

Thus the solution of equation (2), in accordance with the rule stated in verse 61 is $(x, y) = (78, 5\,45 - 15 = 530)$.

3. Alternatively, when the dividend and the interpolator both are abraded by the divisor :

The abridged equation is : $142x - 65 = 143y$.

The minimum solution of $142x + 65 = 143y$ is

$(x, y) = (65, 65)$

Therefore the minimum solution of $142x - 65 = 143y$ is $(x, y) = (143 - 65 = 78, 142 - 65 = 77)$.
Thus the solution of equation (2), in accordance with the rule stated in verse 61 is

$(x, y) = (78, 77 + 78 \times 6 - 15 = 530)$.

2.11 : Rule for solution of a pulveriser when the dividend is negative:

सूत्रम् –

क्षयभाज्ये गुणलब्धी धनवत्साध्ये तु भाज्यतः क्षेपे ।
अल्पे तयोः क्षयं स्यादेकमनल्पे तु ते सकृद्धनगे ॥६२॥

"In the case of a negative dividend find the multiplier and quotient as in the case of its being positive and then subtract them from their respective abraders. One of these results, either the smaller one or the greater one, should be made negative and the other positive."[25] ॥ **62** ॥

उदाहरणम् –

क्षयत्रिंशद्गुणो राशिस्त्रिभिर्युक्तोऽथवोनितः ।
सप्तभक्तो निरग्रः स्यात्तं गुणं वद वेत्सि चेत् ॥३० ॥

Ex. 30: "A number is multiplied by -30. (When) 3 is either added to or subtracted from the product. (The sum or difference) becomes exactly divisible by 7. Tell me the number if you know "[26]॥ **30** ॥

Statement (*Nyāsa*) : Here, the dividend = -30 and the interpolator = 3 and the divisor =7, so the equation will be

$$-30x \pm 3 = 7y \qquad \ldots\ldots\ldots\ldots(1)$$

Solution :
According to the rule 62, in the case of negative dividend, first find the multiplier and quotient as in the case of its being positive and then subtract them from their respective abraders.

25 . Cf. (i). BBi. R. 67(ii). ; (Ii). G.K. ix.R. 28-29. ; HHM. II. p. 122.
26 . GK. IX, Ex. 25. HHM. II. p. 123.

```
7 ) 3 0 ( 4
    2 8
      2 ) 7 ( 3
          6
          1
```

4	4	4× 9+3=39
3	3× 3 + 0 = 9	9
3	3	-
0	-	-

$9 = 7\times 1 + 2$; $39 = 30\times 1 + 9$;

∴ The minimum solution of $7y = 30x + 3$ is $x = 2, y = 9$.

Subtracting these values from the respective abraders, namely 7 and 30, and making one of the remainders negative, we get $x = 5,\ y = -21$ and $x = -5,\ y = 21$ respectively as solutions of

$7y = -30x + 3$ and $7y = -30x - 3$.

In cases like this, Bhāskara observes :

भाजकस्य धनत्वे ऋणत्वे वा लब्धिगुणावेतावेव, परन्तु भाजके भाज्ये वा ऋणगते लब्धेः ऋणत्वं सर्वत्र ज्ञेयम् ।

"Whether the divisor is positive or negative, the numerical values of the quotient and multiplier remain the same: when either the divisor or the dividend is negative , the quotient must always be known to be negative."[27]

अपि च –

क्षयत्रिंशद्गुणः सप्तनवत्या संयुतोनितः ।
सप्ताप्तः शुद्धिमायाति तं गुणं वद मे द्रुतम् ॥३१॥

Ex. 31: "A number is multiplied by -30. (When) 97 is either added to or subtracted from the product. (The sum or

[27] . HHM. II. p. 123.

difference) becomes exactly divisible by 7. Tell me that number quickly. "[28]॥ **31** ॥

Statement (*Nyāsa*) : Here, the dividend = -30 and the interpolator = 97 and the divisor =7, so the equation will be

$$-30x \pm 97 = 7y \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Solution :

According to the rule 62, in the case of negative dividend, first find the multiplier and quotient as in the case of its being positive and then subtract them from their respective abraders.

```
7 ) 3 0 ( 4
    2 8
      2 ) 7 ( 3
          6
          1
```

4	4	4× 291 + 97=1261
3	3× 97 + 0 = 291	291
97	97	-
0	-	-

$291 = 7\times 41 + 4$; $1261 = 30\times 41 + 31$;

∴ The minimum solution of $30x + 97 = 7y$ is $x = 4, y = 31$.

When either the divisor or the dividend is negative , the quotient must always be known to be negative. Therefore, $x = 4, y = -31$ is the minimum solution of $-30x - 97 = 7y$ (when the dividend and the interpolator both are negative).

Subtracting those from their respective abraders i.e., $x = 7 - 4 = 3$; $y = -(30 - 31) = 1$. is the minimum solution of $-30x + 97 = 7y$ (when the dividend is negative and the interpolator positive).

[28] . GK. IX, Ex. 26.

2.12: Rule for the solution of *kuṭṭaka*, when the additive is either zero or an exact multiple of the divisor or the dividend is zero:

सूत्रम् –

हरहृतशुद्धे क्षेपे शून्ये जातेऽथवा गुणः खं स्यात् ।
शून्ये तु भाज्यराशौ हारहृतः क्षेपको लब्धिः ॥६३॥

"Where the additive is either zero or an exact multiple of the divisor, or the dividend is zero, there the multiplier is zero, and the additive as divided by the divisor is the quotient.[29] ‖ 63 ‖

That is, In the equation $ax \pm c = by$,
if $c = 0$ then $x = 0,\ y = \frac{c}{b} = \frac{0}{b} = 0$; or
if(c/b) = an integer, then $x = 0,\ y = \frac{c}{b}$.
Also, if $a = 0$, then, $x = 0$, $y = \frac{c}{b}$,

उदाहरणम् –

को राशिः सप्तहतो नवभिर्युक्तोऽथवोनितः शुद्धम् ।
त्रिभिरुद्धृतः प्रयच्छति भागं तं गुणकमाचक्ष्व ॥३२॥

Ex. 32: "Which is the number when multiplied by 7, and 9 is either added to or subtracted from the product. (The sum or difference) becomes exactly divisible by 3 ? Tell (me) the quotient and the multiplier. "[30]॥ **32** ॥

Statement (*Nyāsa*) : Here, the dividend = 7 and the interpolator = 9 and the divisor =3, so the equation will be

[29]. Cf. (i). L(ASS).Vs.-254. (ii).G.K. ix-30. (iii)BBi. R ॥५८॥63॥.

[30] . GK. IX, Ex. 27.

$$7x \pm 9 = 3y \quad \ldots\ldots\ldots\ldots (1)$$

Solution : Case-1: $7x + 9 = 3y$

Here, since the additive is an exact multiple of the divisor, in accordance with the rule (2.12) stated in the verse 63, the minimum solution, that is, the multiplier is zero (i.e.,$x = 0$), and the additive as divided by the divisor is the quotient. That is, $y = \frac{c}{b} = \frac{9}{3} = 3$.

∴ The general solution of $7x + 9 = 3y$ is :

$$(x,\ y) = (0 + 3t,\ 3 + 7t)$$

Case-2: $7x - 9 = 3y$

In accordance with rule stated in verse 58(ii) , subtracting 0 and 3 from their corresponding abraders 3 and 7, we get the minimum solution as x = 3 - 0 =3, and y = 7-3 =4. Therefore, the general solution in case of 2 with negative interpolator is :
$x = 3 + 3t\ and\ y = 4 + 7\,t$, t being any integer.

अपि च –

को राशिर्नवगुणितः शून्ययुतः पञ्चभिर्हृतः शुद्धिम् ।
गच्छति तं द्राग्राशिं गणकवर ब्रूहि त्वं यदि वेत्सि ॥३३॥

Ex. 33 : "A number is multiplied by 9, (and when) 0 is added (to the product. The sum) becomes exactly divisible by 5. tell the quotient (and the multiplier), if you know."[31]॥ **33** ॥

Statement (*Nyāsa*) : Here, the dividend = 9 and the interpolator =0 and the divisor =5, so the equation will be

$$9x + 0 = 5y \quad \ldots\ldots\ldots\ldots (1)$$

[31] . GK. IX, Ex. 28.

Solution :

Here, since the additive is zero, in accordance with the rule (2.12) stated in the verse 63, the minimum solution, that is, the multiplier is zero (i.e.,$x = 0$), and $y = \frac{0}{5} = 0$.

$\therefore$ The general solution of $9x + 0 = 5y$ is :
$(x, y) = (0 + 5t, 0 + 9t)$
$\therefore$ $x = 5t$ and $y = 9t$, t being any integer.

अपि च –

को राशिः शून्यहतो द्वादशयुक्तो विवर्जितो वऽपि ।
चतुरुधृतो विशुध्यित तं गणक मे कथय ॥३४॥

Ex. 34 : "A number is multiplied by 0, (and when) 12 is either added to or subtracted (from the product.) (The sum or difference becomes) exactly divisible by 4. O mathematician, Tell me the multiplier (and)the quotient)."[32]॥ **34** ॥

Statement (*Nyāsa*) : Here, the dividend = 0 and the interpolator= 12 and the divisor=4, so the equation will be

$$0x \pm 12 = 4y \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Solution :

Here, since the dividend is zero, in accordance with the rule (2.12) stated in the verse 63, the minimum solution, that is, the multiplier is zero (i.e.,$x = 0$), and the additive as divided by the divisor is the quotient. that is, $y = \frac{c}{b} = \frac{12}{4} = 3$.

$\therefore$ The general solution of $0x + 12 = 4y$ is
$(x, y) = (0 + 4t, 3 + 0 \cdot t)$ t being any integer.

[32] . GK. IX, Ex. 29.

The general solution of $0x - 12 = 4y$ is
$x = (4 - 0) + 4t$; $y = (0 - 3) + 0 \cdot t$
or $(x, y) = (4t, -3)$, t being any integer.

Here, Nārāyaṇa comments :

भाज्ये शून्ये लब्धिः एवं सर्वत्रापि अविकृतैव [गुणकोऽपि शून्यानन्तवर्जं सर्वोऽओयभिन्नाङ्कः सम्भवति] ।

"In case the dividend is cipher, the quotient is always constant (and) excluding cipher and infinity, it is possible for the multiplier also to be integral in all cases."

2.13 : Rule for the solution of a given pulveriser, from the solutions of the corresponding Constant Pulveriser :

सूत्रम् –

क्षेपं शुद्धिं रूपं परिकल्प्य तयोः पृथग्गुणाप्ती ये ।
इष्टक्षेपविशुद्ध्या हते स्वहरतक्षिते भवतः ॥६४॥

"The multiplier and quotient determined by supposing the additive or subtractive to be unity, multiplied severally by the desired additive or subtractive and then divided by their respective abraders, (the residues) will be those quantities corresponding to them (i.e., desired interpolators). [33]|| 71 ||

That is, the solution of equation

$$by = ax \pm c$$

in positive integers can be easily derived from that of

$$by = ax \pm 1.$$

[33] (i). MBh. i.45.(ii). BrSpSi. Xviii.9-11. (iii). L(ASS).Vs.257. (iv).BBi..R.॥६६॥ 71 ॥ ; (v). G.K. ix-31.

If $x = m, y = n$ be a solution of the latter equation, we shall have $bn = am \pm 1$.

Then $b(cn) = a(cm) \pm c$.

Hence $x = cm,\ y = cn$ is a solution of the former. The minimum solution will be obtained by abrading the values of x and y thus computed by 'b' and 'a' respectively as indicated before.

2.14 : [Note: Here, Between verses 64 and 65, there occurs in the *Gaṇitakaumudī* a set of four Sūtras followed by six examples dealing with the residual and conjunct pulverisers (*sāgra-kuṭṭaka and saṁśliṣṭa- kuṭṭaka*). To complement the chapter they are given in the appendix I.]

सूत्रम् –

यस्मिन् यस्मिन् कर्मणि यद् यत् परिभाषितं समुदितं च ।
तस्मिन् तस्मिन् कर्मणि त(त्त)त्परिभाषितं भवति ॥६५॥

R.65 : "A definition comming in a particular step in the process is to be used in that step of the process."॥ **65** ॥

2.15 : Rule for solution of Special (astronomical) problems:

त्रैराशिके प्रमाणं हारः परिभाषितोन्मितिर्भाज्यः ।
अवशिष्टमृणक्षेपो या लब्धिस्तत्प्रमाणं स्यात् ॥६६॥
गुणकस्तु पूर्वशेषं तत्पूर्वं पूर्वमेवमपि ।
अनुपातेच्छायामथ ज्ञातायां तत्पलं वाच्यम् ॥६७॥
यो गुणकः सैवेच्छा या लब्धिस्तत्पलं भवति ।

R.66-67$\frac{1}{2}$: "The argument in 'the rule of three is the divisor, the number defining (the constituent) is the dividend (and the remainder is the subtractive, or negative interpolator).The quotient (obtained) happens to be the arguement (and) the multiplier is its requistion. The earlier remainder is the multiplier and and still earlier remainder is the earlier (multiplier) and so on. (Finally) for (the integral number of years) and the unknown requistion as well, the fruit(in the rule of) proportion (i.e., the rule of three) is the dividend. The multiplier obtained happens to be the requistion and the quotient its fruit (i.e., fruit of the requistion). The requistion either so obtained or if given, may as well be used to find the fruit of the demand by the rule of) proportion (i.e., by 'the rule of three')."[34]॥**66-67$\frac{1}{2}$**॥

Explanation and Rationale :

Let a man cover a distance of b *yojanas* in d years. With this rate, let him travel a distance of p *yojans* in q years. Then, $\frac{d}{b} = \frac{q}{p}$

[34] . (i).L(ASS).Vs.258.; (ii).BBi.R.72. (iii). G.K. ix. 37(ii)-39.

or q (in years) = n years, m months, x days, and $y + \frac{c}{b}$ *ghaṭīs,* (say).

Now suppose that neither p nor q is known but only the remainder of *ghaṭīs* after division by b (i.e., only c) is known. The rule gives a process to find p and q both. Since 60 *ghaṭīs* make a day, so 60 is made the dividend. Also, the argument (in 'the rule of three') is b, so b is made the divisor and c, the remainder of *ghaṭīs* after division by b, the subtractive. Thus the equation

$$\frac{60x - c}{b} = y$$

is formed and solved with known b and c.

Now the quotient y, gives the number of *ghaṭīs* and the multiplier x, the remainder of days (after division by b) as $\frac{60x}{b} = y + \frac{c}{b}$, and 60 *ghaṭīs* make a day.

Here, it may be noted that only one solution of the above equation will be valid as $x < b$ and $y < 60$ restricts the valid solution of the equation. Not only that but at all the successive stages of solution of the indeterminate equations formed at those stage in the solution of the problem also. Only one solution of those equation will be valid due to similar restrictions. Now, taking necessary changes in the coefficient of x and in the subtractive quantity and repeating the above process we obtain the number of days, months, and remainder of years (after division by b). Finally the fruit is made the dividend, the remainder of years (after division by b) the subtarctive

as usual, the argument, the divisor. Thus the equation formed is

$$\frac{(dx - c')}{b} = y,$$

where c' is the remainder of years (when divided by b).

So, $$\frac{dx}{b} = y + \frac{c'}{b}.$$

Now, the quotient y gives the number of years. So $y + \left(\frac{c'}{b}\right)$ gives the total time taken in covering the distance. Now, since the man covers a distance of b *yojanas* in d years, so obviously, the multiplier x will give the total distance traversed by the man hence the rule.

The problem can also be solved by 'the rule of three' if either the distance traversed or the time taken is known.

The rule is extensively used in Hindu astronomy[35] for finding (the place of) a planet and the elapsed days, from the remainder of *ghaṭīs* in the planets place.

उदाहाणम् –

पंगुर्योजनषष्टिमेकसहितामब्दैस्त्रिपञ्चाशता
रिङ्गन् क्रामति योजनानि च कियत्सङ् ख्यानियेनासरत् ।
कालेनाशु वदार्य तत्र घटिकाशेषे भवेद् विंशति–
स्तत्सम्वत्सरमासवासरघटीमानानि चेच्छां पृथक् ॥३५॥

Ex.35 : A lame crawls slowly (at the rate of) 61 *yojanas* in 53 years. If the remainder of *ghaṭīs* (after division by

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35 . Shukla, K.S. : Hindu mathematics in the 7th century as found in Bhāskara I's Commentary on the *Āryabhaṭīya, Gaṇita,* Vol. 23, (june, 1972), No.3, 58-59.

61) be 20, O elder, tell quickly the measure of time in years, months, days and *ghaṭīs,* and the distance traversed." ‖ **35** ‖

Solution: Here, 60 *ghaṭīs* =1 day, the remainder of *ghaṭīs* (after division by 61) = 20 and the argument =61. So the equation formed will be, $\frac{(6x-20)}{61} = y$

```
61 ) 60 ( 0
      0
    60 ) 61 ( 1
         60
          1
```

0	0	$0 \times 20 + 20 = 20$
1	$1\times 20 + 0 = 20$	20
20	20	-
0	-	-

$20 = 60 \times 0 + 20$; $20 = 61 \times 0 + 20$

The interpolator being negative, the minimum solutions are $x = 61 - 20 = 41$, and $y = 60 - 20 = 40$.

∴ The general solutions will be, multiplier $x = 41 + 61t$ and the quotient $y = 40 + 60t$, where t is any integer.

However since $x < 61$ and $y < 60$, so its only valid solutions will be, $x = 41 =$ remainder days after division by 61 and $y = 40 =$ number of *ghaṭīs.*

Now, since 30 days= 1 month and remainder of days=41, so the next equation will be ,

$$\frac{30x-41}{61} = y$$

```
61 ) 30 ( 0
      0
    30 ) 61 ( 2
         60
          1
```

0	0	$0\times 82 + 41= 41$
2	$2\times 41 + 0 = 82$	82
41	41	-
0	-	-

$41 = 1 \times 30 + 11$; $82 = 1 \times 61 + 21$

The interpolator being negative, the minimum solutions are $x = 61 - 21 = 40$, and $y = 30 - 11 = 19$.

∴ The general solutions will be, multiplier $x = 40 + 61m$ and the quotient $y = 19 + 30m$, where m is any integer. But as $x < 61$ and $y < 30$ so its valid solution will be $x = 40 =$ number of months after division by 61 and $y = 19 =$ number of days.

Now, since 12 months = 1 year , remainder of months = 40, after division by 61 and $y = 19 =$ number of days, so the next equation formed will be, $\frac{(12x-40)}{61} = y$

```
61 )12 ( 0
     0
   12 ) 61 ( 5
        60
         1
```

0	0	$0\times 200 + 40= 40$
5	$5\times 40 + 0 = 200$	200
40	40	-
0	-	-

$40 = 12 \times 3 + 4$; $200 = 61 \times 3 + 17$

The interpolator being negative, the minimum solutions are $x = 61 - 17 = 44$, and $y = 12 - 4 = 8$.

As, $y < 12$, its only valid solution will be $x = 44 =$ remainder of years after division by 61 and $y = 8 =$ number of months.

Finally, since the fruit = 53 and the remainder of years = 44, so the equation formed will be, $\frac{(53x-44)}{61} = y$.

```
61 ) 53 ( 0
      0
    53 ) 61 ( 1
         53
          8 ) 53 ( 6
              48
               5 ) 8 ( 1
                   5
                   3 ) 5 ( 1
                       3
                       2 ) 3 ( 1
                           2
                           1
```

0	0	0	0	0	0	880
1	1	1	1	1	1012	1012
6	6	6	6	880	880	-
1	1	1	132	132	-	-
1	1	88	88	-	-	-
1	44	44	-	-	-	-
44	44	-	-	-	-	-
0	-	-	-	-	-	-

$880 = 53 \times 16 + 32$; $1012 = 61 \times 16 + 36$

The interpolator being negative, the minimum solutions are $x = 61 - 36 = 25$, and $y = 53 - 32 = 21$.
As $x < 61$, itsonly valid solution willbe
$x = 25$ = the distance traversed, and
$y = 21$ = the number of years
Ths the distance traversed = 25 *yojans* and the time taken= 21 years, 8 months, 19 days and 40 + $\frac{20}{61}$ *ghaṭīs.*

[*Note :* Though the chapter on pulveriser in both the books *Gaṇitakaumudī* and *Bījagaṇitāvatāṃsa* is nearly the same, the following 2 rules, an example and a comment available in this book is not available in the former one.]

2.16 : Rule for the solution of the pulveriser when the solution is fractional:

भिन्नकुट्टके सूत्रम् –

ईप्सितलब्ध्या हारे गुणिते क्षेपोनिते च भाज्याप्ते ।
गुणकारः स्याद्दृढे दृढेऽपि वा कुट् टके भवति ॥६८॥

“Multiply the desired quotient by the divisor. Subtract the additive (from the product). Divide (the difference) by the dividend. (The quotient) happens to be the multiplier in the reduced or unreduced (both types of) pulveriser” ॥ **68** ॥

उदाहरणम् –

कश्चित् त्रिंशद् गुणितो दशसंयुतो द्वादशोद्धृतो यत्र ।
यच्छति शुद्धं भागं कुट्टकगणितज्ञ तं कथय ॥३६॥

Ex.36 : “A number is multiplied by 30. Ten is added to the product. the sum is exactly divisible by 12. Expert in pulveriser, tell that.” ॥ **36** ॥

Statement (*Nyāsa*) : Here, the dividend = 30 and the interpolator =10 and the divisor =12, so the equation will be $30x + 10 = 12y$.

Here, the dividend and the divisor is divisible by 3, but the additive is not divisible by 3. So, according to R. 53 the problem is incorrect.

However , according to Nārāyaṇa [NBi.p.35], if the quotients (the values of y) are taken as 1, 2 and 3, in the equation :

$$x = \frac{by-c}{a} = \frac{12y-10}{30}$$

then the multipliers (the values of x) are obtained as $1/15$, $7/15$ and $13/15$ and so on in many ways, according as the optional numbers are taken.

अपि च –

कुट्टकप्रभवाभावेऽप्यभिन्नलब्ध्य सद गुणो न स्यात् ।
कुट्टकप्राप्तिभावे ऽभीष्टवशात्कदाचिदभिन्नः ॥६९॥

R. 69: "In the pulveriser, the absence of the process (of pulveriser), with integral quotient, the multiplier obtained is not always (so). While obtaining the solution of (such) a pulveriser, due to the desired number, the multiplier obtained is a fraction."

Here, (at the end of this chapter,) Nārāyaṇa comments:

यत्किञ्चित् कुट् टककौशल्यं तत् पुरतोऽनेकवर्णसमीकरणे वक्षे ।

"Certain skills have been told ahead in (the chapter on) simultaneous linear equations."

[Note : Though a part of *Bījagaṇitāvatāṃsa is available, probably its* chapter on (*vaṃa-samatva or anekavarṇa-samīkaraṇa* is now lost.]

इति कुट्टकः

Thus ends the chapter on pulveriser.

अथ वर्गप्रकृतिः

SQUARE-NATURE

The indeterminate quadratic equation

$$Nx^2 \pm c = y^2$$

is called by the Hindus *varga-prakṛti* or *kṛti-prakṛti,* meaning 'Square-Nature'.

According to Datta and singh, the name *varga-prakṛti* originated from the following consideration:

The principle (*prakṛti*) underlying the calculations in this branch of mathematics is to determine a number (or numbers) whose nature (*prakṛti*) is such that its (or their) square (or squares, *varga*) or the simple number (or numbers) after certain specified operations will yield another number (or numbers) of the nature of a square. So the name is very significant.

Technical Terms :

The technical terms ordinarily used by the Hindu algebraists in connection with the Square-nature

$$Nx^2 \pm c = y^2$$

are :

x = *ādya-mūla, kaniṣṭha-pada, hrasva-mūla* (lesser root)
y = *anya-mūla, antya-mūla, jyeṣṭha-pada* (greater root)
N = *prakṛti, guṇa, guṇaka* (multiplier)
c = *kṣepa, prakṣepa, prakṣepaka, kṣipti,* (interpolator)

The interpolator, when negative, is sometimes distinguished as 'subtractive' (*śodhaka*). The positive interpolator is then called 'the additive'.

Brahmagupta (628 A.D.), Jayadeva (early 11th century A. D.), Bhāskara II (1150 A.D.) and Nārāyaṇa (1356 A.D.) etc. have given rules for the solution of such equations.

***Varga-Prakṛti in Bījagaṇitāvatāṃsa* :**

The third section of *Bījagaṇitāvatāṃsa* deals with the indeterminate equation of the second degree (*varga-prakṛti*) i.e., $Nx^2 \pm c = y^2$. In it first of all an obvious solution of equation by supposition has been given in (3.1) in verse70. After that, the additive composition has been dealt with in R.71-73(i) and the subtractive Composition in R. 73(i)-74(i). The case when the interpolator is the product of a square with an integer has been treated in 74(ii)-75(i). In R.75(ii)-76(i), a method for obtaining a rational solution of the equation has been given. R76(ii)gives a method to obtain an infinite number of solutions from one such solution of the equation.

Cakravāla (Cyclic Method of solution of the equation) has been dealt eith in R. 77-80. Solutions of equations $Nx^2 - k^2 = y^2$, $Mn^2x^2 \pm c = y^2$ and $m^2x^2 \pm c = y^2$, have been given in R.81, R.82 and R.83 respectively. A rule to find an infinite number of solutions of $Nx^2 \pm c = y^2$, with the help of one of its solution and a solution of $Nx^2 + 1 = y^2$, have been given in R.84-85. Finally, at the end of the section a method to find the approximate value of a quadratic surd is treated in R.86.

३. वर्गप्रकृतिः

वर्गप्रकृतौ सूतमार्यासप्तकम् –

ह्रस्वमभीष्टं मूलं तद्वर्गः प्रकृतिसङ् गुणो युक्तः ।
हीनो वा येन कृतिः स्यात्तस्मा(त्त)त्पदं ज्येष्ठम् ॥७०॥

3.1 : Definitin, and an obvious solution by supposition :

"An optionally chosen number is taken as the lesser root (*hrasva-mūla*). Multiply its square by the multiplier (*prakṛti*). Add (the interpolator) to or subtract (it) from (the product). The (result) is the square, whose root is the greater root (*jyeṣṭha -mūla*).[1] ॥ **70** ॥

Observations[2] :

The terms 'lesser root' and 'greater root' do not appear to be accurate and happy. For if $x = m$, $y = n$ be a solution of the equation $Nx^2 + c = y^2$, m will be less than n, if N and c are both positive. But if they are of opposite signs, the reverse will sometimes happen. Therefore, in the latter case, where $m > n$, it will be obviously ambiguous to call m the lesser root, as was the practice in later Hindu algebra.

For instance, take the following example (BBi. verse 97)

$$13x^2 - 13 = y^2.$$

One solution of it is given by Bhāskara as $x = 1$, $y = 0$; so that here the lesser root is greater than the greater root. The same is the case in the solution $x = 2$, $y = 1$ in the example (BBi. verse 98)

[1] . (i). BBi. R. 75. ; (ii). SiTvi. xiii. vs.209. Cf. HHM. II. P. 144.
[2] . HHM. II. p. 144. f.n. 5. & p145. f.n. 1

$$-5x^2 + 21 = y^2.$$

Brahmagupta gives the example (BrSpSi. Xviii - 77)

$$3x^2 - 800 = y^2,$$

which has a solution ($x = 20,\ y = 20$) where the two roots are equal.

This defect in the prevalent terminology was noticed by Kṛṣṇa (1580). He explains it thus :

अन्वर्थाश्चैतास्संज्ञाः । यत्र तु क्षेपवियोगात्कुत्रचिज्ज्येष्ठपदं ह्रस्वपदादल्पं भवति तत्रापि भावनया ह्रस्वपदादधिकमेव भवति ॥

These terms are significant. Where the greater root is sometimes smaller than the lesser root owing to the interpolator being negative, there also it becomes greater than the lesser root after the application of the Principle of Composition.

For example, by composition of the solution (1, 0) of the equation $13x^2 - 13 = y^2$, with the solution $\left(\frac{3}{2}, \frac{11}{2}\right)$ of the equation $13x^2 + 1 = y^2$, we obtain, after Bhāskara II, a new solution $\left(\frac{11}{2}, \frac{39}{2}\right)$ of the former , in which the greater root is greater than the lesser root.

Similarly, by composition of the solution (2, 1) of the equation $-5x^2 + 21 = y^2$ with the solution $\left(\frac{1}{3}, \frac{2}{3}\right)$ of the equation $-5x^2 + 1 = y^2$, we get a new solution (1, 4) of the former satisfying the same condition.

The earlier terms, 'the first root' (*ādya-mūla*) for the value of x and 'the second root' for the value of y, are quite free from ambiguity. Their use is found in the algebra of Brahmagupta (628). The later terms appear in the works of his commentator Pṛthudakasvāmī (860).

Before proceeding to the general solution of the Square-nature Brahmagupta has established two important lemmas.

3.2 : Brahmagupta's Lemma-I ('*Bhāvana*' or Principle of Composition)[3] :

" Of the square of an optional number multiplied by the *guṇaka* and increased or decreased by another optional number, (extract) the square-root. (Proceed) twice. The product of the first roots multiplied by the *guṇaka* together with the product of the second roots will give a (fresh) second root ; the sum of their cross products will be a (fresh) first root. The corresponding interpolator will be equal to the product of the (previous) interpolators."[4]

That is to say,

If $x = \alpha,\ y = \beta$ be a solution of the equation

$$Nx^2 + k = y^2,$$

and $x = \alpha',\ y = \beta'$ be a solution of

$$Nx^2 + k' = y^2,$$

then according to the above,

$$x = \alpha\beta' \pm \alpha'\beta,\ y = \beta\beta' \pm N\alpha\alpha'$$

is a solution of the equation

$$Nx^2 + kk' = y^2.$$

In other words, if

$$N\alpha^2 + k = \beta^2,$$
$$N{\alpha'}^2 + k' = {\beta'}^2,$$

then

$$N(\alpha\beta' \pm \alpha'\beta)^2 + kk' = (\beta\beta' \pm N\alpha\alpha')^2. \quad \ldots\ldots\text{(I)}$$

In particular, taking $\alpha = \alpha',\ \ \beta = \beta'$ and $k = k'$, Brahmagupta finds from a solution $x = \alpha,\ y = \beta$ of the equation

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3 . HHM. II. p. 146-147.

4 . मूलं द्विधेष्टवर्गाद्गुणकगुणादिष्टयुतविहीनाच्च ।
आद्यवधोगुणकगुणः सहान्त्यघातेन कृतमन्त्यम् ॥ 64 ॥
वज्रवधैक्यं प्रथमं प्रक्षेपः क्षेपवधतुल्यः । [Br.Sp.Si.xviii. 64-65(i)]

$$Nx^2 + k = y^2,$$

a solution $x = 2\alpha\beta,\ y = \beta^2 + N\alpha^2$ of the equation

$$Nx^2 + k^2 = y^2.$$

That is, if $N\alpha^2 + k = \beta^2,$

then , $N(2\alpha\beta)^2 + k^2 = (\beta^2 + N\alpha^2)^2$ ... ... (II)

This result will be hereafter called *Brahmagupta's corollary.*

The above results are called by the technical name, *Bhāvana* (demonstration or proof, meaning anything demonstrated or proved, hence theorem, lemma ; the word also means composition or combination).

They are further distinguished as *Samāsa-bhāvanā* (addition Lemma or Additive Composition) and *Antara Bhāvanā* (Subtraction Lemma or Subtractive Composition). Again, when the *Bhāvana* is made with two equal sets of roots and interpolators, it is called *Tulya Bhāvanā* (Composition of Equals) and when with two unequal sets of values, *Atulya Bhāvanā* (Composition of Unequals). *Tulya Bhāvanā* (Composition of Equals) has been named Brahmagupta's Corollary by B.B.Datta and A.N. Singh (HHM, II, p.147)

Brahmagupta's Lemma-I has been described by Nārāyaṇa thus :

3.3 : *Bhāvanā* (Principle of Composition) :

ह्रस्वज्येष्ठक्षेपान् क्रमशस्तेषामधो न्यसेत्तांस्तु ।
अन्यान्येषां न्यासस्तस्य भवेद् भावना नाम ॥७१॥

"Set down the lesser root, the greater root (and) the interpolator. Set down below them the same or another set (of similar quantities), in order. Placing in such a way, (and the principle by which numerous roots can be obtained from them,) is called *Bhāvanā* (Principle of Composition). ॥ **71** ॥

3.3.1: *Samāsa-bhāvanā* (Addition Lemma or Additive Composition) :

वज्राभ्यासौ ह्रस्वज्येष्ठ(क)योः संयुतिर्भवेद्ध्रस्वम् ।
लघुघातः प्रकृतिहतो ज्येष्ठवधेनान्वितो ज्येष्ठम् ॥७२॥
क्षिप्त्योर्घातः क्षेपः स्याद् ...

"The sum of the two cross-products of the two lesser and the two greater roots happens to be a lesser root. Multiply the product of the two lesser roots by the *prakṛti* . (The product so obtained), added to the product of two greater roots, is a greater root. (In that equation), the product of two interpolators is the interpolator."[5]

[5] . Cf. (i) .Br.Sp.Si. xviii.64-65. ; (ii).BBi. R. ॥ ७१॥77-78 ॥
(iii). SiTVi. Xiii.210-214. HHM. II. pp. 147- 148. ;

3.3.2: *Antara Bhāvanā* (Subtraction Lemma or Subtractive Composition) :

...वज्राभ्यासयोर्विशेषो वा ।
ह्रस्वं लघ्वोर्घातः प्रकृतिघ्नो ज्येष्ठयोश्च वधः ॥७३॥
तद्विवरं ज्येष्ठपदं क्षेपः क्षिप्त्योः प्रजायते घातः ।

"Alternatively, the difference of the two cross-products is a lesser root; and the difference obtained when the product of the two lesser roots multiplied by the *prakṛti* is subtracted from the product of the two greater roots will be a greater root. (Here also), the interpolator is the product of the two (previous) interpolators."[6]

Explanation : That is, if $x = \alpha,\ y = \beta$ be a solution of the equation $Nx^2 + k = y^2$, and $x = \alpha',\ y = \beta'$ be a solution of $Nx^2 + k' = y^2$, then according to the above,

Step.1. N α β k
α' β' k'

Step.2. $(\alpha\beta' + \alpha'\beta)^2 + kk' = (\beta\beta' + N\alpha\alpha')^2. \ldots\ldots (1)$

Step.3. $(\alpha\beta' - \alpha'\beta)^2 + kk' = (\beta\beta' - N\alpha\alpha')^2. \ \ldots\ (2)$

Proof : The proof of *Bhāvanā* (Principle of Composition) has been given by Kṛṣṇa substantially as follows :

We have

$$N\alpha^2 + k = \beta^2,$$
$$N\alpha'^2 + k' = \beta'^2.$$

[6]. Cf. (i) .Br.Sp.Si. xviii.64-65. ; (ii).BBi. R. ॥ ७१ ॥ 77-78 ॥
(iii). SiTVi. Xiii.210-214. HHM. II. pp. 147- 148.

Multiplying the first equation by β'^2, we get

$$N\alpha^2\beta'^2 + k\beta'^2 = \beta^2\beta'^2.$$

Now, substituting the value of the factor β'^2 of the interpolator (k) from the second equation, we get

$$N\alpha^2\beta'^2 + k\left(N\alpha'^2 + k'\right) = \beta^2\beta'^{2'},$$

or $$N\alpha^2\beta'^2 + Nk\alpha'^2 + kk' = \beta^2\beta'^2.$$

Again, substituting the value of k from the first equation in the second term of the left-hand side expansion, we have

$$N\alpha^2\beta'^2 + N\alpha'^2(\beta^2 - N\alpha^2) + kk' = \beta^2\beta'^2,$$

or $$\mathrm{N}(\alpha^2\beta'^2 + \alpha'^2\beta^2) + kk' = \beta^2\beta'^2 + N^2\alpha^2\alpha'^2.$$

Adding $\pm\ 2N\alpha\beta\alpha'\beta'$ to both sides, we get

$$N(\alpha\beta' \pm \alpha'\beta)^2 + k\,k' = (\beta\beta' \pm N\alpha\alpha')^2.$$

Brahmagupta's Corollary follows at once from the above by putting $\alpha' = \alpha,\ \beta' = \beta$ and $k = k'$.

Kṛṣṇa has observed that when it is desired to derive roots of a Square-nature, larger in value, one should have recourse to the Addition Lemma (पदयोर्महत्वेऽपेक्षिते समासभावना) and for smaller roots one should use the Subtraction Lemma (पदयोर्लघुत्वेऽभीप्सितेन्तरभावना).

3.4 : Brahmagupta's Lemma-II [7]:

"On dividing the two roots (of a Square-nature) by the square-root of its additive or subtractive, the roots for the interpolator unity (will be found)."[8]

That is to say, if $x = \alpha,\ y = \beta$ be a solution of the equation

$$Nx^2 + k^2 = y^2,$$

then $x = \frac{\alpha}{k}\ ,\ y = \frac{\beta}{k}$ is a solution of the equation

$$Nx^2 + 1 = y^2.$$

This rule has been restated in a different way thus :

" If the interpolator is that divided by a square then the roots will be those multiplied by its square root."[9]

That is, suppose the Square-nature to be

$$Nx^2 \pm p^2 d = y^2,$$

so that its interpolator $p^2 d$ is exactly divisible by the square p^2. Then, putting therein $u = \frac{x}{p}\ ,\ v = \frac{y}{p}$, we derive the equation

$$Nu^2 \pm d = v^2,$$

whose interpolator is equal to that of the original Square-nature divided by p^2. It is clear that the roots of the original equation are p times those of the derived equation.

7. HHM. II. p. 150-151.
8 प्रक्षेपशोधकहृते मूले प्रक्षेपके रूपे ॥ 65 ॥ – [Br.Sp.Si. xviii. 65(ii)]
9 वर्गच्छिने क्षेपे तत्पदगुणिते तदा मूले ॥७०॥ –[Br.Sp.Si. xviii. 70(ii)]

Brahmagupta's Lemma-II has been described by Nārāyaṇa thus :

3.5: Rule for Solution of $Nx^2 \pm p^2k = y^2$ or $Nx^2 \pm (k/p^2) = y^2$:

ईप्सितवर्गविहृतः क्षेपः क्षेपः पदे तदिष्टाप्ते ॥७४॥
गुणिते वा तन्मूले गुणिते मूले तदा भवतः ।

"(If) the interpolator (of a Square-nature) divided or multiplied by the square of an optional number, be the interpolator (of another Square-nature), (then) the two roots (of the former) divided or multiplied, (as the case may be) by that optional number, are the roots (of the other Square-nature)."[10] ॥ **74 -75(i)** ॥

That is, in general, we have, if $x = \alpha,\ y = \beta$ be a solution of the equation

$$Nx^2 \pm k = y^2,$$

then, $x = \alpha/m\ ,\ y = \beta/m$ is a solution of the equation

$$Nx^2 \pm k/m^2 = y^2;$$

and $x = n\alpha\ ,\ y = n\beta$ is a solution of the equation

$$N \pm n^2 k = y^2,$$

where *m* , *n* are arbitrary rational numbers.

This rule has been stated in slightly different words by Nārāyaṇa (1356) (cf. *NBi, I,* R.74-74½; *GK*. x. 5b-6a.) and Kamalākara (1658) (*SiTVi,* Xiii. 215) .

Jñānarāja (1503) simply observes :

"If the interpolator (of a square-nature) be divided by the square of an optional number then its root will be divided by that optional number."

10. BBi. R.॥७२॥ 79 ॥ ; GK, X, R. 5(ii)-6(i); HHM. II. p. 151.

Brahmagupta's Lemmas were rediscovered and recognised as important by Euler in 1764 and by Lagrange in 1768.

3.6 : Solution of $Nx^2 + 1 = y^2$

It was recognised that the most fundamental equation of the Square-nature, that is of the class $Nx^2 \pm c = y^2$ is

$$Nx^2 + 1 = y^2,$$

where N is a non-square integer.

Due to a mistake on the part of Euler, this equation : $Nx^2 + 1 = y^2$, was given the name Pell's equation. Pell just referred to this problem in a book on algebra that he wrote. The equation was nearly solved by Brahmagupta (628), and was improved first by Jayadeva (c.1000) and next by Bhāskara II (1150), and it is therefore fitting that this equation be called the Brahmagupta- Bhāskara equation.[11]

The complete theory underlying the solution was expounded by Lagrange in 1767, and rests on the theory of continued fractions.

[11]. Srinivasiengar C.N. : *The History of Ancient Indian Mathematics* world press, Calcutta, 1967. p. 110.

3.6.1 : General solution of the Square-Nature :

It is clear from Brahmagupta's Lemma-I that when two solutions of the Square-nature,

$$Nx^2 + 1 = y^2,$$

are known, any number of other solution can be found. For, if two solutions be (a, b) and (a', b'), then two other solutions will be

$$x = ab' \pm a'b, \quad y = bb' \pm Naa'.$$

Again, composing this solution with the previous ones, we shall get other solutions. Further, it follows from Brahmagupta's Corollary that if (a, b) be a solution of the equation, another solution of it is $(2ab, \ b^2 + Na^2)$. Hence, in order to obtain a set of solutions of the Square-nature it is necessary to obtain only one solution of it. For after having obtained that, an infinite number of other solutions can be found by the repeated application of the Principle of Composition.

3.6.2 : Rational solution (Tentative Method) :

In order to obtain a first solution of $Nx^2 + 1 = y^2$ the Hindus generally suggest the following tentative method : Take an arbitray small rational number α, such that its square multiplied by the *guṇaka* N and increased or diminished by a suitable chosen rational number k will be an exact square. In other words, we shall have to obtain empirically a relation of the form

$$N\alpha^2 \pm k = \beta^2,$$

where α, β, k are rational numbers. This relation is usually called as the 'Auxiliary Equation'. Then by Brahmagupta's Corollary, we get from it the relation

$$N(2\alpha\beta)^2 + k^2 = (\beta^2 + N\alpha^2)^2,$$

or $$N\left(\frac{2\alpha\beta}{k}\right)^2 + 1 = \left(\frac{\beta^2+N\alpha^2}{k}\right)^2.$$

Hence, one rational solution of the equation $Nx^2 + 1 = y^2$ is given by $x = \frac{2\alpha\beta}{k}$, $y = \frac{\beta^2 + N\alpha^2}{k}$.

3.6.3 : Sritpati's Rational Solution (Direct Method):

Sripati (1039) has shown how a rational solution of the Square-nature can be obtained more easily and directly without the intervention of an auxiliary equation. He says :

"Unity is the lesser root. Its square multiplied by the *prakṛti* is increased or decreased by the *prakṛti* combined with an (optional) number whose square-root will be the greater root. From them will be obtained two roots by the principle of Composition."[12]

If m^2 be a rational number optionally chosen, we have the identity $N \cdot 1^2 + (m^2 - N) = m^2$,

or $N \cdot 1^2 - (N - m^2) = m^2$.

Then, applying Brahmagupta's corollary to either, we get

$$N(2m)^2 + (m^2 \sim N)^2 = (m^2 + N)^2;$$

$$\therefore\ N\left(\frac{2m}{m^2\sim N}\right)^2 + 1 = \left(\frac{m^2+N}{m^2\sim N}\right)^2.$$

Hence $$x = \frac{2m}{m^2\sim N}, \quad y = \frac{m^2+N}{m^2\sim N}$$

where m is any rational number, is a solution of the equation $Nx^2 + 1 = y^2$.

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12 रूपं कनीयःपदमस्य वर्गे हते प्रकृत्या वियुते वा ।
क्षिप्त्यापदं यच्च बृहत्पदं तत् ताभ्यां पदे भावनया त्वनन्ते ॥
[Si.Se.xiv.33.]

The above solution has been given by Nārāyaṇa as follows:

3.7 : Sritpati's Rational Solution in Nārāyaṇa's words :

इष्टकृतिगुणकशेषोद्धृतं तदिष्टं द्विसंगुणं भवति ॥७५॥
ह्रस्वं मूलं च ततो रूपक्षेपेण साधयेज्ज्येष्ठम् ।

"Twice an optional number, divided by the difference between the square of that optional number and the *prakṛti,* happens to be the lesser root (of a Square-nature when unity is the additive). From that with unity as the additive, the greater root should be obtained."[13] **॥75(ii)-76(i) ॥**

That is, If m be an optional number, it is stated that $\frac{2m}{m^2\sim N}$ is a lesser root of $Nx^2 + 1 = y^2$.

Then , substituting that value of x in the equation, we get

$$y^2 = N\left(\frac{2m}{m^2\sim N}\right)^2 + 1$$

$$= \left(\frac{m^2+N}{m^2\sim N}\right)^2$$

Hence the greater root is $y = \frac{m^2+N}{m^2\sim N}$.

The same solution will be obtained by assuming

$$y = mx - 1.$$

That is, $Nx^2 + 1 = m^2x^2 - 2mx + 1$

or $(N - m^2)x^2 + 2mx = 0$

---

13. Cf. (i).BBi. R.॥७३॥ 80-81 ; (ii).GK.. X. 6(ii)-7(i). ; (iii).SiTVi., Xiii.216. HHM. II. p.154.

$$\therefore x = \left(\frac{2m}{N-m^2}\right)$$

Kṛṣṇa points out that it can also be found thus :

$$4Nm^2 = (m^2 + N)^2 - (m^2 \sim N)^2, \text{ identically.}$$

$$\therefore \quad 4Nm^2 + (m^2 \sim N)^2 = (m^2 + N)^2,$$

$$\text{or} \qquad \therefore \; N\left(\frac{2m}{m^2 \sim N}\right)^2 + 1 = \left(\frac{m^2+N}{m^2 \sim N}\right)^2.$$

This rational solution of the Square-nature is due to Śrīpati (1039). It was rediscovered in Europe by Brouncker(1657).

Infinite solutions :

3.8: Rule to find Infinite number of solutions :

तुल्यातुल्यपदानां भावनयाऽनन्तमूलानि ॥७६॥

"By the Principle of Composition of equal as well as unequal sets of roots, (will be obtained) an infinite number of roots."[14] ॥ **76(ii)** ॥

Datta and Singh pointed out that, 'the modern historians of mathematics are incorrect in stating that Fermat (1657) was the first to assert that the equation $Nx^2 + 1 = y^2$, where N is a non-square integer, has an unlimited number of solutions in integers. The existence of an infinite number of integral solutions was clearly mentioned by Hindu algebraists long before Fermat.'

[14] . Cf. BBi. R. ॥७३॥ 81 ॥ ; Cf. GK. x. 6b -7a.
HHM. II. p.150. (Also see f.n. 2).

उदाहरणम् –

अष्टाहता यस्य कृतिः सरूपा स्यान्मूलदा ब्रूहि सखे तमाशु ।
एकादशघ्ना यदि वा कृतिः का वर्गत्वमेत्येकयुता विचिन्त्य ॥३७॥

Ex. 37 : O friend, tell me quickly (the number) whose square (when multiplied by 8 and then added to 1 yields a square-root, or thinking well (tell the number) whose square if multiplied by 11 (and then) added to 1 becomes square."[15] ॥ **37** ॥

Statement (*Nyāsa*) : Solve (i). $\mathbf{8x^2 + 1 = y^2}$ **,**
(ii). $\mathbf{11x^2 + 1 = y^2}$ **.**

Solution :

Case (i). Here the equation is $\mathbf{8x^2 + 1 = y^2}$ ... $(\mathbf{1})$

Let the lesser root (= l) be 1. Then the greater root (= g) for the interpolator (= i) will be 3.

Now the statement for the composition as stated in verse 71 is,

N	l	g	i
N = 8	1	3	1
	1	3	1

By the principle of Additive-Composition (of equals) stated in verse 72-73(i) :

$N = 8,\; l = (2\cdot 1\cdot 3)\;;\; g = 8\cdot 1^2 + 3^2\;;\; i = 1\cdot 1$
$\quad = 6 \qquad = 17 \qquad = 1$

Statemen for the composition for the next case

N = 8	1	3	1
	6	17	1

Bythe principle of Additive-Composition (of unequals)

N = 8, $l = (1\cdot 17 + 3\cdot 6)\;;\; g = (8\cdot 1\cdot 6 + 3\cdot 17)\; i = 1\cdot 1$
$\quad = 35 \qquad = 99 \qquad = 1$

[15] . HHMM. II. pp.155-157.

Again statement for composition for the next case will be

N = 8	1	3	1
	35	99	1

Bythe principle of Additive-Composition (of unequals)

N=8 ; $l = (1 \cdot 99 + 3 \cdot 35) = 204$; $g = (8 \cdot 1 \cdot 35 + 3 \cdot 99) = 577$; and so on . "Thus an infinite number of roots" may be found.

Alternatively, take the equation $8x^2 + 4 = y^2$. Here, $l = 2$, $g = 6$ when $k = 4$. Now by rule stated in verses 74(ii)-75(i) we get , $l = 1, g = 3$ when $k = 1$ as the solution of $8x^2 + 1 = y^2$.

Now, again an infinite number of solutions of the equation $8x^2 + 1 = y^2$ can be obtained by the Principle of Composition of Equalas as well as Unequals.

Again according to Sritpati's Rational Solution stated in verses 75(ii)-76(i) , if the optional number (=m)=3, then

$x = \left(\frac{2m}{N \sim m^2}\right) = \frac{2\times 3}{8\sim 3^2} = 6$, $y = \frac{m^2 + N}{m^2 \sim N} = \frac{3^2+8}{3^2\sim 8} = 17$ for k=1 .

If $m = 5, x = \frac{2\times 5}{8\sim 5^2} = \frac{10}{17}$, $y = \frac{5^2+8}{5^2\sim 8} = \frac{33}{17}$ for $k = 1$.

Now from these two solutions, by Additive Composition, we get,

N = 8	6	17	1
	$\frac{10}{17}$	$\frac{33}{17}$	1

$l = 6 \cdot \frac{33}{17} + 17 \cdot \frac{10}{17} = \frac{368}{17}$; $g = 8 \cdot 6 \cdot \frac{10}{17} + 17 \cdot \frac{33}{17} = \frac{1041}{17}$ for k=1

By Subtractive Composition, we get,

$l = 6 \cdot \frac{33}{17} - 17 \cdot \frac{10}{17} = \frac{28}{17}$; $g = 17 \cdot \frac{33}{17} - 8 \cdot 6 \cdot \frac{10}{17} = \frac{81}{17}$ for k=1.

Case (ii). $\mathbf{11x^2 + 1 = y^2}$ **.**

Consider the equation $11x^2 - 2 = y^2$.

Wehave, $N = 11$ $l = 1$, $g = 3$, $when\, k = -2$.

Now the statement for the composition as stated in verse 71 is,

N	l	g	i
N = 11	1	3	-2
	1	3	-2

By the principle of Additive-Composition (of equals) stated in verse 72-73(i) :

$N = 11$, $l = (2 \cdot 1 \cdot 3) = 6$; $g = 11 \cdot 1^2 + 3^2 = 20$; for $i = -2 \cdot -2 = 4$

Now by rule stated in verses 74(ii)-75(i) we get , $l = \frac{6}{2} = 3$, $g = \frac{20}{2} = 10$ when $k = 1$ as the solution of $11x^2 + 1 = y^2$.

Again, by the principle of Additive-Composition (of equals) stated in verse 72-73(i) :

N = 11	3	10	1
	3	10	1

$N = 11$, $l = (2 \cdot 3 \cdot 10) = 60$; $g = 11 \cdot 3^2 + 10^2 = 199$; for $i = 1 \cdot 1 = 1$.

Alternatively, take the equation $11x^2 + 25 = y^2$. We have, $N = 11$, $l = 8$, $g = 27\, for\, k = 25$.

Now by rule stated in verses 74(ii)-75(i) we get ,

$l = \frac{8}{5}$, $g = \frac{27}{5}$, $for\, k = 1$.

Also, by rule stated in verses 74(ii)-75(i) we get ,

$l = 3$, $g = 10\, for\, k = 1$.

From the Additive Composition, we now get

N = 11	3	10	1
	$\frac{8}{5}$	$\frac{27}{5}$	1

$l = \frac{80}{5} + \frac{81}{5} = \frac{161}{5}$; $g = \frac{270}{5} + \frac{11}{1} \times \frac{8}{5} \times \frac{3}{1} = \frac{534}{5}$; $i = 1 \times 1 = 1$.

and again "an infinite number of roots" may be obtained . Also, from Subtractive Composition we get

$$l = \frac{81}{5} - \frac{80}{5} = \frac{1}{5}\,; \quad g = \frac{270}{5} - \frac{11}{1} \times \frac{8}{5} \times \frac{3}{1} = \frac{6}{5}\,; \quad i = 1 \times 1 = 1.$$

and again "an infinite number of solutions" can be obtained.

Again according to Sritpati's Rational Solution stated in verses 75(ii)-76(i) , if the optional number (=m)=3, then

$$x = \left(\frac{2m}{N \sim m^2}\right) = \frac{2 \times 3}{11 \sim 3^2} = 3\,, y = \frac{m^2 + N}{m^2 \sim N} = \frac{3^2 + 11}{3^2 \sim 11} = 10 \text{ for k=1 .}$$

If $m = 5, x = \frac{2 \times 5}{11 \sim 5^2} = \frac{5}{7}, y = \frac{5^2 + 11}{5^2 \sim 11} = \frac{18}{7}$ for $k = 1.$

Now from these two solutions, by Additive Composition, we get, N = 8

	3	10	1
	$\frac{5}{7}$	$\frac{18}{7}$	1

$l = 3 \cdot \frac{13}{7} + 10 \cdot \frac{5}{7} = \frac{104}{7}; g = 10 \cdot \frac{18}{7} + 11 \cdot 3 \cdot \frac{5}{7} = \frac{345}{7}$ for k=1

By Subtractive Composition, we get,

$l = 3 \cdot \frac{18}{7} + 10 \cdot \frac{5}{7} = \frac{4}{7}; g = 10 \cdot \frac{18}{7} - 11 \cdot 3 \cdot \frac{5}{7} = \frac{15}{7}$; k=1. and so on.

इति वर्गप्रकृतिः

3.9 : Solution of $Nx^2 + 1 = y^2$ in positive integers :

The aim of the Hindus was to obtain solutions of the Square-nature in positive integers ; so its first solution must be integral. But neither the tentative method of Brahmagupta nor that of Sripati is of much help in this direction, for they do not always yield the desired result. These authors, however, discovered that if the interpolator of the auxiliary equation in the tentative method be ± 1, ± 2 or ± 4, an integral solution of the equation $Nx^2 + 1 = y^2$ can always be found (BrSpSi. xviii. 67-68; SiSe. xiv-32).

चक्रवालम्

CYCLIC METHOD

The most fundamental step in Brahamagupta's method for the general solution in positive integers of the equation

$$Nx^2 + 1 = y^2$$

where N is a non-square integer, is to form an auxiliary equation of the kind

$$Na^2 + k = b^2,$$

where a, b are positive integers and $k = \pm 1, \pm 2 \; or \; \pm 4$. For, from that auxiliary equation, by the Principle of Composition , applied repeatedly whenever necessary, one can derive, one positive integral solution of the original Square-nature. And thence, again by means of the same principle, an infinite number of other solutions in integers can be obtained.

How to form an auxiliary equation of this type (i.e.,with $k = \pm 1, \pm 2 \; or \; \pm 4$) was a problem which could not be solved completely and satisfactorily by Brahmagupta.

But, first Jayadeva[1] and next Bhāskara II succeeded in evolving a very simple and elegant method by means of which one can derive an auxiliary equation having the required interpolator $\pm 1, \pm 2, or \pm 4$, simultaneously with its two integral roots, from another auxiliary equation empirically formed with any simple integral value of the

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[1] K.S. Shukla : ' Ācārya Jayadeva, The Mathematician' *Ganita, 5(1).*June 1954, pp1-20.

interpolator, positive or negative. This method is called by the technical name *Cakravāla* or the "Cyclic Method."

The purpose of the Cyclic Method has been defined by Bhāskara II thus [cf.verse 86(i)] :

चतुर्द्व्येकयुतावेवमभिन्ने भवतः पदे ।

By this method, there will appear two integral roots corresponding to an equation with $\pm 1, \pm 2$ *or* ± 4 as interpolator.

3.10 : Bhāskara's Lemma [2]:

The Cyclic Method of Bhāskara II is based upon the following Lemma :

If $$Na^2 + k = b^2,$$

where $a,\ b,\ k$ are integers, k being positive or negative, then,

$$N\left(\frac{am+b}{k}\right)^2 + \frac{m^2-N}{k} = \left(\frac{bm+Na}{k}\right)^2,$$

where m is an arbitrary whole number.

The rationale of this Lemma is simple :

We have

$$Na^2 + k = b^2,$$

and $$N \cdot 1^2 + (m^2 - N) = m^2, \text{ identically.}$$

Then by Brahmagupta's Lemma, we get

$$N(am + b)^2 + k(m^2 - N) = (bm + Na)^2.$$

$$\therefore\ N\left(\frac{am+b}{k}\right)^2 + \frac{m^2-N}{k} = \left(\frac{bm+Na}{k}\right)^2.$$

2 . HHM. II. p. 162.

चक्रवालम्

3.11 : Cyclic Method (*Cakravāla*) of Nārāyaṇa :

एकद्विचतुःक्षेपसाधनाय चक्रवाले करणसूत्रमार्याचतुष्टयम् –

ह्रस्वज्येष्ठक्षेपान् भाज्यप्रक्षेपभाजकान् कृत्वा ।
कल्प्यो गुणो यथा तद्वर्गात् संशोधयेत् प्रकृतिम् ॥७७॥

प्रकृतेर्गुणवर्गे वा विशोधिते जायते तु यच्छेषम् ।
तत् क्षेपहृतं क्षेपो गुणवर्गविशोधिते व्यस्तम् ॥७८॥

लब्धिः कनिष्ठमूलं तन्निजगुणकाहतं वियुक्तं च ।
पूर्वाल्पपदपरक्षिप्त्योर्घातेन जायते ज्येष्ठम् ॥७९॥

प्रक्षेपशोधनेष्वप्येकद्विचतुर्ष्वभिन्नमूले स्तः ।
द्विचतुःक्षेपपदाभ्यां रूपक्षेपाय भावना कार्या ॥८०॥

"Making the lesser root, greater root and interpolator (of a square-nature) the dividend, addend and divisor (respectively of a pulveriser), the (indeterminate) multiplier of it should be determined in the way described before. The *prakṛti* being subtracted from the square of that or the square of the multiplier being subtracted from the *prakṛti,* the remainder divided by the (original) interpolator is the interpolator (of a new Square-nature); and it will be reversed in sign in case of subtraction of the square of the multiplier. The quotient (corresponding to that value of the multiplier) is the lesser root (of the new Square-nature) ; and that multiplied by the multiplier and diminished by the product of the previous lesser root and

(new) interpolator will be its greater root. By doing so repeatedly will be obtained two integral roots corresponding to the interpolator $\pm 1, \pm 2 \; or \; \pm 4$. In order to derive integral roots for the additive unity from those answering to the interpolator $\pm 2 \; or \; \pm 4$, the Principle of Composition (should be adopted)."[3] ॥ **77-80** ॥

Explanation : To solve, $Nx^2 + 1 = y^2$, ...(1)
first of all an auxiliary equation $Na^2 + k = b^2$ ... (2)
is formed, in which a, b, k are simple integers, relatively prime, k being positive or negative and N is known from (1).

Now a pulveriser, $\frac{am+b}{k}$ = an integer = p, is formed and general value of indeterminate multiplier m is obtained from it. A suitable value of $m = (m_1 \; or \; n)$ **[preferably, though not necessarily, that value which makes $|m^2 - N|$ least]** is taken and another auxiliary equation $N{a_1}^2 + k_1 = {b_1}^2$ ....(3)
is formed such that

$$a_1 = \left\{\frac{(am_1+b)}{k}\right\}; \; k_1 = \frac{({m_1}^2 \sim N)}{K}, \text{ and } b_1 = (a_1 m_1 - k_1 a).$$

Now the process is repeated with (3) in place of (2) and a similar auxiliary equation $N{a_2}^2 + k_2 = {b_2}^2$ is obtained. Repetition of the process, again and again, leads to an auxiliary equation in which the interpolator is ± 1, ± 2, $or \; \pm 4$ which gives the solution of equation (1) by the use of the Principle of Composition.

[3]. Cf. (i).BBi..R.॥७५॥83-86॥.(ii)G.K.Ch.X.R.8-11; HHM. II.p. 162-166.

Rationale :

(1). *Nārāyaṇa's Lemma* : The Cyclic Method of Nārāyaṇa is based upon the following Lemma : If $Na^2 + k = b^2$ where a, b, k are simple integers, relatively prime, k being positive or negative, then

$$N{a_1}^2 + k_1 = {b_1}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\text{where, } a_1 = \left\{\frac{(an+b)}{k}\right\} \ldots\ldots\ldots\ldots\ldots (4)$$

$$k_1 = \frac{(n^2 \sim N)}{K} \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots (5)$$

$$b_1 = (a_1 n - k_1 a) \quad \ldots\ldots\ldots \; \ldots (6)$$

and n a suitable integral value of indeterminate multiplier.

We have,

$Na^2 + k = b^2$. Also $N \cdot 1^2 + (n^2 - N) = n^2$, identically.
Now by the application of Additive Composition to the two relations, we get

$$\begin{array}{ccc} a & b & k \\ 1 & n & (n^2 - N) \end{array}$$

$$N(an + b)^2 + k(n^2 - N) = (Na + bn)^2$$

or [dividing by k^2]

$$N\left\{\frac{(an+b)}{k}\right\}^2 + \left\{\frac{(n^2-N)}{k}\right\} = \left\{\frac{(Na+bn)}{k}\right\}^2 \quad \ldots\ldots (7)$$

Now, substituting in the expression $\frac{(Na+bn)}{k}$ the value of b obtained from (4), weget

$$\frac{(Na+bn)}{k} = \frac{Na+(a_1 k - an)n}{k}$$

$$= a_1 n - \left(\frac{n^2-N}{k}\right) a$$

$= (a_1 n - k_1 a)$, since, from (5) $\frac{n^2-N}{k} = k_1$

$= b_1$ since , from (6) $(a_1 n - k_1 a) = b_1$

Therefore, from (7) we get (3) due to (4),(5), and (6).

(2). k_1, b_1, are integral if a_1 is integral.

From $Na_1{}^2 + k_1 = b_1{}^2$ it is clear that b_1 will be integral if k_1 is also so, as N and a_1 are integral.

From above, we have,

$$\frac{(Na+bn)}{k} = a_1 n - \left(\frac{n^2-N}{k}\right) a = (a_1 n - k_1 a) = b_1,$$

or $$a_1 n - \left(\frac{n^2-N}{k}\right) a = b_1$$

or $$k(a_1 n - b_1) = a(n^2 - N)$$

or $$\frac{k(a_1 n - b_1)}{a} = (n^2 - N)$$

Therefore, $k(a_1 n - b_1)$ is an integer.

Now, k and a have no common factor, so a must divide $a_1 n - b_1$.

$\therefore$ $\frac{(a_1 n - b_1)}{a} = \frac{(n^2-N)}{k} = k_1$, an integer.[4]

(3). Recursive Character :

The *Cakravāla* process of solving $Nx^2 + 1 = y^2$, has intimate relation with the continued fractions. Several authors such as , Ayyangar, A.A.K. and Majumdar,

4 . HHM, II, pp.165-166.

P.K.[5] etc. have treated the topic. Selenius, C.[6] and Paramanand Singh,[7] have explained the process (including its recursive character), the former with the help of semi-regular continued fraction and the latter with the help of regular continued fraction. Since we will need it again in explaining the next rule, let us define it here and state some of the results associated with it.

Let $a_n, b_n,$ and r_n be defined by the relations,[8]

$$\frac{\sqrt{N}+b_n}{r_n} = a_n + \frac{\sqrt{N}-b_{n+1}}{r_n} = a_n + \frac{r_{n+1}}{\sqrt{N}+b_{n+1}}$$

for values $n = 1, 2, 3, \ldots\ldots$; a_n where a_n is an integer such that, $a_n, < \frac{\sqrt{N}+b_n}{r_n} < a_{n+1}$;

$$b_{n+1} = a_n r_n - b_n \quad \text{and} \quad r_n r_{n+1} = N - (b_{n+1})^2$$

leading to

$$\sqrt{N} = a_1 + \frac{1}{a_2 +}\ \frac{1}{a_3 +} \ldots\ldots \frac{1}{a_{c-1} +}\ \frac{1}{a_c +}\ \frac{1}{2\,a_1 +}\ \frac{1}{a_2 +}\ \ldots.$$

so that the number of elements in the reccurring cycle is c.

5 . (i). Majumdar, P. K. : *Gaṇita Kaumudī* and the continued fraction, *IJHS*, Vol. 13, No. 1, (1978), 1-5

(ii). Majumdar, P. K. : indeterminate Quadratic Equation in Indian and Central Asian Context, *Interaction between India and Central Asian Science and Technology in Medieval Times,* Vol, I, 271-277.

6 . C.Selenius, : Rationale of the *Cakravāla* process of Jayadeva and Bhāskara II, Historia Mathematica, II (1975), 167-184.

7 . Singh, Paramanand : *Varga-Prakṛti* –the *Cakravāla* Method of of its Solution and Regular Continued Fractionss, *IJHS*, Vol.19. No.1, 1983, 9.

8 . Barnard ,S. and Child, J. M. : *Higher Algebra,* Macmillan and Col. Ltd., London, 1952, 530-531.

Let $\frac{p_r}{q_r}$ be rth convergent for the continued fraction.

The whole process of *Cakravāla* has been explained with the help of regular continued fraction. It has been shown that all quantities in the *Cakravāla* process have simple counter parts in the regular continued fraction expansion of $\sqrt{N}$. Thus r_n is the interpolator, q_n , the lesser root and p_n, the greater root. Moreover, not only the recursive character of the *Cakravāla* process has been proved which at once establishes that the process will always lead to the solution of (1), but Principle of Composition and Brahmagupta's another Lemma have also been explained in that light.[9]

4. Principle of Composition :

Though the *Cakravāla* process always leads to the solution of $Nx^2 + 1 = y^2$, ...(1)
due to its recursive character, the use of the Principle of Composition at a stage when any one out of -1, ± 2, ± 4 is obtained as an interpolator, brilliantly shortens the process.

(i). If $k = -4$, use of Brahmagupta's Corallary (i.e., *Tulya Bhāvanā*) will immediately lead to a relation with 4 as additive, as the use of the corollary make lesser root (and the greater root as well) divisible by 4.

9 . Ref. 7, 1 - 17.

(ii). If $k = 4$ and the roots are even, use of of Brahmagupta's Corallary (i.e., *Tulya Bhāvanā*) will immediately lead to a solution of (1) as it will make lesser root (and the greater root as well) divisible by the interpolator 16.

(iii). If $k = 4$ and the roots are odd :

Let the relation be $Na^2 + 4 = b^2$ . . . .(2)

By Brahmagupta's Corallary (i.e., *Tulya Bhāvanā*), we get

$$N(2ab)^2 + 4 \cdot 4 = (b^2 + Na^2)^2$$
$$= \{b^2 + (b^2 - 4)\}^2 \quad \text{due to (2)}$$
$$= \{2(b^2 - 2)\}^2$$

or $N(ab)^2 + 4 = (b^2 - 2)^2$ . . . .(3)

By the Composition of (2) and (3), we have

N	a	b	4
	ab	$b^2 - 2$	4

$$N\{a(b^2 - 2) + ab^2\}^2 + 4 \cdot 4 = \{Na^2 b + b(b^2 - 2)\}^2$$
$$N\{ab^2 + ab^2 - 2a\}^2 + 4 \cdot 4 = \{(b^2 - 4)b + b(b^2 - 2)\}^2$$
$$N\{2a(b^2 - 1)\}^2 + 4 \cdot 4 = \{b^2 b - 4b + b^2 b - 2b\}^2$$
$$= \{2b^2 b - 6b\}^2$$
$$= \{2b(b^2 - 3)\}^2.$$

This gives the solution of (1), as a and b both are odd, and so both the sides are divisible by 16.

(iv). If $k = \pm 2$ Brahmagupta's Corallary (i.e., *Tulya Bhāvanā*) will immediately lead to the solution of (1) due to the interpolator will now become 1

(v). If $\boldsymbol{k = -1}$ Brahmagupta's Corallary (i.e., *Tulya Bhāvanā*) will lead to the solution of (1).

Earliest treatment of *Cakravāla* method for the solution of *Varga-Prakṛti* [10] is by Ācārya Jayadeva (early 11th century A.D.). The treatment is found in the form of quotations from his work in the *Sundarī* commentary (1073 A.D.) by Udayadivākara on the *Laghu-Bhāskarīya* of Bhāskara I. According to C.Selenius, *Cakravāla* method is considered tobe the absolute climax of the old Indian mathematics and so of all Oriental mathematics. This method anticipated the European methods more than thousand years. No European performances in the whole field of algebra at a time much later than Bhāskara's, nay upto our time, equalled the marvellous complexity and ingenuity of *Cakravāla.*[11]

10 . Shukla. K.S. : Ācārya Jayadeva, the mathematician, *Gaṇita,* Vol.5No.1 June 1954. 1-20.

11 . See Ref. 6

उदाहरणम् –

कस्त्र्युत्तरेण गुनितोऽत्र शतेन वर्गः
सैकः कृतित्वमुपयाति वदाऽऽशु तं मे ।
को वा त्रिवर्जितशतेन हतस्तु वर्गो
रूपान्वितः कृतिगतो भवति प्रचक्ष्व ॥३८॥

Ex.38: "Tell me quickly that (number) which (when) multiplied by 103, (the product) added to 1, becomes a square, or tell (the number) which (when) multiplied by 97, the product added to 1, becomes a square."[12]**॥38॥**

Statement (*Nyāsa*) : (i). $\boldsymbol{103x^2 + 1 = y^2}$,
(ii). $\boldsymbol{97x^2 + 1 = y^2}$ **.**

Solution : Case (i). $\boldsymbol{103x^2 + 1 = y^2}$

Let the auxiliary equation be $103(1)^2 - 3 = (10)^2$.
Here, the lesser root $(= a) = 1$,
the interpolator $(= k) = -3$,
and the greater root $(= b) = 10$.
So, according to the Lemma stated in verse 77,

$$103\left(\frac{m+10}{-3}\right)^2 + \frac{m^2-103}{-3} = \left(\frac{10m+103}{-3}\right)^2.$$

the pulveriser formed is $\frac{1\cdot m+10}{-3} = an\ integer = s$

$$\begin{array}{rl} 3\ ) & 1\ (\ 0 \\ & \underline{0} \\ & 1 \end{array}$$

0	$0 \times 10 + 0 = 0$
10	10
0	-

$10 = 3 \times 3 + 1$

12 . HHM II, pp.168-171

The general solution of the pulveriser is

$m = (3-1) + (-3t)$, t being any integer.

Taking $t = -3$, we have, $m = 11$.

Now, by the rule (Lemma stated in verses 77-79),

the next auxiliary equation formed will be

$$103{a_1}^2 + k_1 = {b_1}^2$$

where, $a_1 = \left(\frac{m+10}{-3}\right) = \left(\frac{11+10}{-3}\right) = -7$

$$k_1 = \left(\frac{m^2-103}{-3}\right) = \left(\frac{11^2-103}{-3}\right) = -6$$

and $b_1 = (a_1 m - k_1 a) = \big(-7 \times 11 - (-6)\big) = -71$.

So, the next auxiliary equation formed will be

$$103(-7)^2 + (-6) = (-71)^2$$

Now according to Nārāyaṇa :

ऋणधनमूलयोरुत्तरे कर्मणि क्रियमाणे न विशेषः ।

“For further working there is no difference between negative and positive toots”. So taking the roots as positive, the auxiliary equation becomes :

$$103(7)^2 - 6 = (71)^2.$$

Again by the Lemma,

$$103\left(\frac{7n+71}{-6}\right)^2 + \frac{n^2-103}{-6} = \left(\frac{71n+103\cdot 7}{-6}\right)^2.$$

The next pulveriser will be $\frac{7n+71}{-6} = an\ integer$

6	)	7	(	1
		6		
		1		

1	1× 71 + 0 = 71
71	71
0	-

$71 = 6 \times 11 + 5$; $6-5=1$,

Its solution is: $n = -6t + 1$, where t is any integer.

Taking $t = -1$, we get $n = 7$.

Now the auxiliary equation, $103{a_2}^2 + k_2 = {b_2}^2$ will be

$$103 \cdot (20)^2 + 9 = (203)^2,$$

as $a_2 = \left(\frac{7\cdot n+71}{-6}\right) = \left(\frac{7\cdot 7+71}{-6}\right) = -20$

$$k_2 = \left(\frac{n^2-103}{-6}\right) = \left(\frac{7^2-103}{-6}\right) = 9$$

and $b_2 = (a_2 n - k_2 a_1) = \big(-20 \times 7 - (9 \times -7)\big) = -203$.

Next, we have

$$103\left(\frac{20p+203}{9}\right)^2 + \frac{p^2-103}{9} = \left(\frac{203p+103\cdot 20}{9}\right)^2.$$

The next pulveriser is $\frac{20p+203}{9} = an\ integral\ number$

```
9  )  2  0  (  2
      1  8
         2  )  9  (  4
               8
               1
```

2	2	2× 812 + 203 = 1827
4	4 × 203 + 0 = 812	812
203	203	-
0	-	-

$812 = 90 \times 9 + 2$;

∴ General Solution of the pulveriser is

$p = 2 + 9t$; t being any integer.

Taking $t = 1$, we get $p = 11$. On taking this value we find

$$103 \cdot (47)^2 + 2 = (477)^2,$$

Applying the Principle of Composition of Equals, we get

$$103(2\cdot 47\cdot 477)^2 + 2^2 = (477^2 + 103\cdot 47^2)^2,$$

or $\qquad 103\cdot(44838)^2 + 4 = (455056)^2.$

Hence $\qquad 103\cdot(22419)^2 + 1 = (227528)^2.$

This gives $x = 22419$, $y = 227528$ as a solution of (i).

Case (ii). $\boldsymbol{97x^2 + 1 = y^2}$ **.**

Let the auxiliary equation be $97\cdot 1^2 + 3 = 10^2$,

Here, the lesser root $(= a) = 1$,

the interpolator $(= k) = 3$,

and the greater root $(= b) = 10$.

So, according to the Lemma stated in verse 77,

$$97\left(\frac{m+10}{3}\right)^2 + \frac{m^2-97}{3} = \left(\frac{10m+97}{3}\right)^2,$$

The pulveriser formed is $\frac{m+10}{3} = an\ integer,$

3) 1 (0
$\underline{0}$
1

	0 × 1 + 10 = 10
10	10
0	-

$$10 = 3 \times 3 + 1\ .$$

The general solution of the pulveriser is

$m = (3-1) + (3t),\quad t$ being any integer.

Taking $t = 3$, we have, $m = 11$.

Now, by the rule (Lemma stated in verses 77-79),

the next auxiliary equation formed will be

$$103{a_1}^2 + k_1 = {b_1}^2$$

where, $a_1 = \left(\frac{m+10}{3}\right) = \left(\frac{11+10}{3}\right) = 7$

$$k_1 = \left(\frac{m^2-97}{3}\right) = \left(\frac{11^2-97}{3}\right) = 8$$

and $b_1 = (a_1 m - k_1 a) = \big(7 \times 11 - (8\cdot 1)\big) = 69.$

So, the next auxiliary equation formed will be

$$97\cdot(7)^2 + (8) = (69)^2.$$

Again by the Lemma,

$$97\cdot\left(\frac{7n+69}{8}\right)^2 + \frac{n^2-97}{8} = \left(\frac{69n+97\cdot 7}{8}\right)^2.$$

The next pulveriser will be $\frac{7n+69}{8} = an\ integer$

8) 7 (0
0
7) 8 (1
7
1

0	0	0× 69 + 69 = 69
1	1× 69 + 0 = 69	69
69	69	-
0	-	-

$$69 = 8 \times 8 + 5.$$

Its solution is: $n = 8t + 5$, where t is any integer.

Taking $t = 1$, we get $n = 13$.

Now the auxiliary equation, $103{a_2}^2 + k_2 = {b_2}^2$ will be

$$97\cdot(20)^2 + 9 = (197)^2,$$

as $\qquad a_2 = \left(\frac{7\cdot n+69}{8}\right) = \left(\frac{7\cdot 13+69}{8}\right) = 20$

$$k_2 = \left(\frac{n^2-97}{8}\right) = \left(\frac{13^2-97}{8}\right) = 9$$

and $b_2 = (a_2 n - k_2 a_1) = \big(20 \times 13 - (9 \times 7)\big) = 197.$

Next, we have

$$97\left(\frac{20p+197}{9}\right)^2 + \frac{p^2-97}{9} = \left(\frac{197p+97\cdot 20}{9}\right)^2.$$

The next pulveriser is $\frac{20p+197}{9} = an\ integral\ number$

9) 20 (2
18
2) 9 (4
8
1

2	2	2× 788 + 197 = 1773
4	4× 197 + 0 = 788	788
197	197	-
0	-	-

$778 = 87 \times 9 + 5$

The general solutin of the pulveriser is :

$p = 9t + 5$, where t is any integer.

Taking $t = 1$, we get $p = 14$.

We have noted earlier that in taking the value of the multiplier p, **Nārāyaṇa does not expressly state that p should be so chosen as will make $|p^2 - N|$ the least.** Here, we note that $p = 5$ will make $|5^2 - 97|$, the least and not $|14^2 - 97|$, but Nārāyaṇa has taken $p = 14$ and not $p = 5$.

On taking $p = 14$ we find

$$97 \cdot (53)^2 + 11 = (522)^2.$$

Whence

$$97\left(\frac{53q+522}{11}\right)^2 + \frac{q^2-97}{11} = \left(\frac{522q+97\cdot 53}{11}\right)^2.$$

The next pulveriser is $\frac{53q+522}{11} = an\ integer.$

11) 53 (4
44
9) 11 (1
9
2) 9 (4
8
1

4	4	4	12528
1	1	2088+ 522= 2610	2610
4	4× 522 = 2088	2088	
522	522	-	
0	-	-	

$2610 = 237 \times 11 + 3$

Its general solution is $= (11-3) + 11t$, t is any integer.

Taking $t = 0$, we have $q = 8$. Now, the next auxiliary equation will be, $97 \cdot (86)^2 - 3 = (847)^2$.

Next we find

$$97\left(\frac{86r+847}{-3}\right)^2 + \frac{r^2-97}{-3} = \left(\frac{847r+97\cdot 53}{-3}\right)^2.$$

3) 86 (28
84
2) 3 (1
2
1

28	28	28 × 847 + 847= 24563
1	1× 847 + 0 = 847	847
847	847	-
0	-	-

$847 = 282 \times 3 + 1$

Solution of the pulveriser is $r = -3t + 1$, t is any integer.

Taking $t = -3$, we obtain $r = 10$. With this value of r, the next auxiliary equation will be,

$$97 \cdot (569)^2 - 1 = (5604)^2.$$

Applying the Principle of Composition of Equals, we get

$$97 \cdot (2 \cdot 569 \cdot 5604)^2 + 1 = (5604^2 + 97 \cdot 569^2)^2,$$

or $$97 \cdot (6377352)^2 + 1 = (62809633)^2.$$

Thus $x = 6377352$, $y = 62809633$ is a solution of

$$97x^2 + 1 = y^2.$$

Solution of *varga- prakṛti* of the form: $Nx^2 - k^2 = y^2$:

सूत्रमार्या –

रूपविशुद्धौ प्रकृतिः कृतियोगः स्यान्न चेत् खिलं तु तदा ।
अखिलप्रकृतौ प्राग्वत् साध्ये मूलेऽल्पकानल्पे ॥८१॥

3.12 : Rule for solution of *varga- prakṛti*, when unity is the subtractive, and the solution of the problem is not to be impossible :

"In the case of unity as the subtractive, the multiplier must be the sum of two squares. Otherwise, the solution is impossible. If the multiplier be such that the solution is possible, the greater and the lesser roots should be obtained by the method stated earlier"[13] ॥ **81** ॥

Thus it has been said that a rational solution of

$$Nx^2 - 1 = y^2,$$

and consequently of

$$Nx^2 - k^2 = y^2$$

is not possible unless N is the sum of two squares.

For, if $x = \frac{p}{q}$, $y = \frac{r}{s}$ be a possible solution of the equation, we have

$$N\left(\frac{p}{q}\right)^2 - k^2 = \left(\frac{r}{s}\right)^2,$$

or $$N = \left(\frac{qr}{ps}\right)^2 + \left(\frac{qk}{p}\right)^2.$$

The equation can be solved by *Cakravāla* method.

As is clear by the solutions of the examples given by Nārāyaṇa, if $N = m^2 + n^2$, two rational solutions of

[13] . Cf. (i). BBi, R. ॥७७॥ 88 ॥; (ii). GK. X. 12. ; HHM. II. pp. 178.

$Nx^2 - 1 = y^2$ will be, $x = \frac{1}{m}, y = \frac{n}{m}$ and $x = \frac{1}{n}, y = \frac{m}{n}$ and so, two rational solutions of ,

$Nx^2 - k^2 = y^2$ will be $x = \frac{k}{m}, y = \frac{kn}{m}$ and $x = \frac{k}{n}, y = \frac{km}{n}$.

According to Nārāyaṇa : " चक्रवालेनाभिन्ने मूलानि "

That is, "Integral roots (should be obtained) by *Cakravāla* Cclic Method)."

Besides these, Bhāskara II has given several other methods for the solution of such equations.[14]

उदाहरणम् –

कस्त्रयोदशनिघ्नश्च वर्गो व्येकः पदप्रदः ।
को वर्ग एकषष्टिघ्नो निरेको मूलदो वद ॥३९॥

Ex. 39: Tell, which number (when) multiplied by 13, (the product) lessened by 1, yields a square or which number (when) multiplied by 61, (the product lessened by 1 becomes a square."[15] ॥ **39** ॥

Statement (*Nyāsa*) : That is to solve the equations

(i). $13x^2 - 1 = y^2$,

(ii). $61x^2 - 1 = y^2$.

Solution :

Ex. (i). $13x^2 - 1 = y^2$,

Method-1:

An obvious solution of $13x^2 - 4 = y^2$ is $x = 1$, $y = 3$.

[14] . BBi. ‖ 77-79‖ 88-90 ‖ , HHM, II, p. 178-181.

[15] . HHM, II, pp.179-181. also see p.168.

Then dividing by 4 (in accordance with rule stated in verse 74-75(i), we get a solution of $13x^2 - 1 = y^2$ as $\left(\frac{1}{2}, \frac{3}{2}\right)$.

Method-2:

Again, since an obvious solution of $13x^2 - 9 = y^2$ is $x = 1,\ y = 2$.

Then dividing by 9 (in accordance with rule stated in verse 74-75(i), we get a solution of $13x^2 - 1 = y^2$ as $\left(\frac{1}{3}, \frac{2}{3}\right)$.

Method-3:

From these fractional roots, we may derive integral roots by the Cyclic Method. Since

$$13\left(\frac{1}{2}\right)^2 - 1 = \left(\frac{3}{2}\right)^2,$$

we have, by Bhāskara's Lemma,

$$N\left(\frac{am+b}{k}\right)^2 + \frac{m^2 - N}{k} = \left(\frac{bm+Na}{k}\right)^2$$

where, $a = \frac{1}{2}$, $b = \frac{3}{2}$, $k = -1$, $N = 13$ and m being an indeterminate multiplier,

$$13\left(\frac{\frac{m}{2}+\frac{3}{2}}{-1}\right)^2 + \frac{m^2 - 13}{-1} = \left(\frac{\frac{3m}{2}+\frac{13}{2}}{-1}\right)^2,$$

or
$$13\left(\frac{m+3}{-2}\right)^2 + \frac{m^2 - 13}{-1} = \left(\frac{3m+13}{-2}\right)^2.$$

The suitable value of m which will make $\frac{(m+3)}{2}$ an integer and $|m^2 - 13|$ minimum is 3. So that we have

$$13.(3^2) + 4 = 11^2$$

From this again we get the relation

$$13\left(\frac{3n+11}{4}\right)^2 + \frac{n^2 - 13}{4} = \left(\frac{11n + 13\times 3}{4}\right)^2.$$

The appropriate value of the indeterminate multiplier in this case is $n = 3$. Substituting this value, we have

$$13.(5^2) - 1 = 18^2.$$

Hence an integral solution of our equation $13x^2 - 1 = y^2$ is $(5,\ 18)$.

Method-4: In the equation $13x^2 - 1 = y^2$

Since $13 = 2^2 + 3^2$, i.e., $m = 2,\ n = 3$, and $k = -1$; then according to the method implied in the solution of examples given by Nārāyaṇa, that is, if $N = m^2 + n^2$ in the equation $Nx^2 - k^2 = y^2$, then two rational solutions of it are : $x = \frac{k}{m}, y = \frac{kn}{m}$ and $x = \frac{k}{n}, y = \frac{km}{n}$.

∴ the two rational solutions are $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\left(\frac{1}{3}, \frac{2}{3}\right)$.

Ex. (ii). $61x^2 - 1 = y^2$.

Method-1:

An obvious solution of $61x^2 - 36 = y^2$ is $(x,\ y) = (1,\ 5)$.

Then according to the rule stated in verses 74-75(i).

$(x,\ y) = \left(\frac{1}{6}, \frac{5}{6}\right)$ is a solution of $61x^2 - 1 = y^2$.

Similarly,again an obvious solution of $61x^2 - 25 = y^2$ is $(x,\ y) = (1,\ 6)$. Then according to the rule stated in verses 74-75(i). $(x,\ y) = \left(\frac{1}{5}, \frac{6}{5}\right)$ is a solution of $61x^2 - 1 = y^2$.

Method-2: In the equation $61x^2 - 1 = y^2$

Since $61 = 5^2 + 6^2$, i.e., $m = 5,\ n = 6$, and $k = -1$; then according to the method implied in the solution of examples given by Nārāyaṇa, that is, if $N = m^2 + n^2$ in

the equation $Nx^2 - k^2 = y^2$, then two rational solutions of it are : $x = \frac{k}{m}, y = \frac{kn}{m}$ and $x = \frac{k}{n}, y = \frac{km}{n}$.

$\therefore$ the two rational solutions are $\left(\frac{1}{5}, \frac{6}{5}\right)$ and $\left(\frac{1}{6}, \frac{5}{6}\right)$.

Method -3 : Cyclic Methodfor integral roots :

Here we start with the auxiliary equation

$$61 \cdot 1^2 + 3 = 8^2.$$

By Lemma, we have

$$61\left(\frac{m+8}{3}\right)^2 + \frac{m^2 - 61}{3} = \left(\frac{8m+61}{3}\right)^2 \quad \ldots . \textbf{(1)}$$

Now the solution of $\frac{m+8}{3}$ = an integer, is $m = 3t + 1$.

Putting $t = 2$, we get the value $m = 7$

On subtituting this value in (1), it becomes

$$61\left(\frac{7+8}{3}\right)^2 + \frac{7^2 - 61}{3} = \left(\frac{8.7+61}{3}\right)^2$$

$$61 . 5^2 - 4 = 39^2.$$

Dividing out by 4, we get

$$61.\left(\frac{5}{2}\right)^2 - 1 = \left(\frac{39}{2}\right)^2 \ldots \ldots \ldots \textbf{(2)}$$

By the principle of Composition of Equals, we have

$$61.\left(2.\frac{5}{2}.\frac{39}{2}\right)^2 + 1 = \left\{\left(\frac{39}{2}\right)^2 + 61.\left(\frac{5}{2}\right)^2\right\}^2$$

$$61.\left(\frac{195}{2}\right)^2 + 1 = \left(\frac{1523}{2}\right)^2 \ldots \ldots \ldots \ldots \textbf{(3)}$$

Combining (2) and (3) , by the Principle of Composition of unequals $61(3805)^2 - 1 = (29718)^2$.

$\therefore$ $(x, y) = (3805, 29718)$ is a solution of $61x^2 - 1 = y^2$.

अपि च –

वर्गः पञ्चगुणः कश्चित् चतुर्भिः संयुतः कृतिः ।
षट्त्रिंशताऽथवा युक्तः शतयुक्तोऽथवा भवेत् ॥४०॥

Ex. 40 : “Which (is the number whose) square multiplied by 5, (the product) either added to 4 , or to 36, or to 100 happens to be square.” ॥ **40** ॥

Statement (*Nyāsa*) :

(1). $5x^2 + 4 = y^2$

(2). $5x^2 + 36 = y^2$

(3). $5x^2 + 100 = y^2$

Solution : $(x, y) = (1, 3)$ is an obvious solution of (1). According to ythe rule stated in verses 74(ii) - 75(i) :

“If the interpolator of a Square-nature divided or multiplied by the square of an optional number, be the interpolator of another Square-nature, then the two roots of the former divided or multiplied, as the case may be by that optional number, are the roots of the other Square-nature.”

Considering the optional number as 3, we obtain the the solution of equation (2). Therfore multiplying the roots of the equation (1) by 3 we get the roots of (2). Therfore, $(x, y) = (1 \times 3 = 3, 3 \times 3 = 9)$ is a solution of (2).

Similarly considering the optional number as 5 , we obtain the the solution of equation (3). Therfore multiplying the roots of the equation (1) by 5 we get the roots of (3). Therfore, $(x, y) = (1 \times 5 = 5, 3 \times 5 = 15)$ is a solution of (3).

Solution of *varga- prakṛti* of the form: $Mn^2x^2 \pm k = y^2$:

सूत्रम् –

प्रकृतिरभीप्सितवर्गोद्धृता यथा शुद्धिमेति यल्लब्धम् ।
कल्प्यो गुणः कनिष्ठं छेदनमूलोत्धृतं भवति ॥८२॥

3.13 : Rule for finding the roots , where the *prakṛti* is exactly divisible by a square-number :

"Divide the multiplier (of a Square-nature) by an arbitrary square number so that there is left no remainder. Take the quotient as the multiplier (of another Square-nature). The lesser root (of the reduced equation) divided by the square-root of the divisor will be the lesser root (of the original equation)."[1] ॥ **82** ॥

That is to say, suppose the equation to be

$$Mn^2x^2 \pm c = y^2, \ \ldots\ldots \ \ldots\ldots(1)$$

so that the multiplier (i.e., coefficient of x^2) is divisible by n^2. Putting $nx = u$, we get

$$Mu^2 \pm c = y^2. \ \ldots\ldots \ \ldots\ldots(2)$$

Then clearly the first root of (1) is equal to the first root of (2) divided by n. The corresponding second root will be the same for both the equations.

उदाहरणम् –

द्वासप्ततिप्रगुणिता कृतिरेकयुक्ता
मूलप्रदा भवति मे वद मित्र शीघ्रम् ।
पञ्चांश्केन गुणितोऽप्यथवा सरूपो
वर्गः कृतित्वमुपयाति सखे विचिन्त्य ॥४१॥

[1] . Cf. (i). BBi. R. ॥८२॥ 93॥. ; (ii) GK.Ch.x.R.13.; HHM, II, p. 176.

Ex. 41 : "O friend, tell me quickly (the number whose) square multiplied by 72, (the product) added to 1 happens to be a square, or (tell number whose) square multiplied by $1/5$ (the product) added to 1 happens to be square." ॥ **41** ॥

Statement (*Nyāsa*) : (i). $\mathbf{72x^2 + 1 = y^2}$;

(ii). $\left(\frac{1}{5}\right)\mathbf{x^2 + 1 = y^2}$.

Solutin : Case (i). $\mathbf{72x^2 + 1 = y^2}$

This equation can be written in the following form:

$$8 \cdot 3^2x^2 + 1 = y^2$$

Putting $3x = u$, the equation reduces to $8u^2 + 1 = y^2$
an obvious solution of which is $(u, y) = (1, 3)$
Then by the rule stated in verse 82,
$(x,\ y) = \left(\frac{1}{3},\ 3\right)$ is a solution of $72x^2 + 1 = y^2$.

Case (ii). $\left(\frac{1}{5}\right)\mathbf{x^2 + 1 = y^2}$

This equation can be written in the following form:

$5 \cdot \left(\frac{1}{5}\right)^2 x^2 + 1 = y^2$

Putting $\frac{1}{5}x = v$, the equation reduces to $5v^2 + 1 = y^2$
an obvious solution of which is $(v, y) = (4,\ 9)$
Then by the rule stated in verse 82,
$(x,\ y) = \left(\frac{4}{(1/5)} = 20,\ 9\right)$ is a solution of $\frac{1}{5}x^2 + 1 = y^2$.

By repeated application of the Principle of Additive and Subtractive Compositions, an infinite number of roots may be found .

Solution of *varga- prakṛti* of the form: $a^2x^2 \pm k = y^2$:

वर्गगतायां प्रकृतौ सूत्रम् –

क्षिप्तिरभीष्टविभक्ता द्विधा तदिष्टोनसंयुता दलिता ।
आद्या प्रकृतिपदाप्ता क्रमशोऽल्पानल्पमूले ते ॥८३॥

3.14 : Rule finding the roots of *varga- prakṛti* when the coefficient (*prakṛti*) is a square number :

"The intepolator divided by an optional number is set down at two places ; the quotient is diminished (at one place) and increased (at the other) by that optional number and then halved. The former is again divided by the square-root of the multiplier. (The quotients) are respectively the lesser and greater roots."[2] ॥ **83** ॥

That is, to find the solution of the equation

$$a^2x^2 \pm c = y^2.$$

Let, m be an arbitrary number, then according to the rule stated above

Step-1 : $\frac{\pm c}{m}$ $\qquad$ $\frac{\pm c}{m}$

Step-2 : $\frac{\pm c}{m} - m$ $\qquad$ $\frac{\pm c}{m} + m$

Step-3 : $\frac{1}{2}\left(\frac{\pm c}{m} - m\right)$ $\qquad$ $\frac{1}{2}\left(\frac{\pm c}{m} + m\right)$

Step-4 : $x = \frac{1}{2a}\left(\frac{\pm c}{m} - m\right)$ $\qquad$ $y = \frac{1}{2}\left(\frac{\pm c}{m} + m\right)$

are the solutions of $a^2x^2 \pm c = y^2$.

[2] . Cf. (i). BBi. R. ॥८४॥ 95 ॥; (ii). GK. x. R. 14. ; HHM. II. p.177.

The rationale of the above solution has been given by the commentators Sūryadāsa and Kṛṣṇa substantially as follows :

$$\pm c = y^2 - a^2x^2$$
$$= (y - ax)(y + ax).$$

Assume $(y - ax) = m$ ……… (i)

m being an arbitrary rational number.

Then $y + ax = \frac{\pm c}{m}$ ……… (ii)

From (i) and (ii) by the rule of concurrence, we get

$$x = \frac{1}{2a}\left(\frac{\pm c}{m} - m\right),$$

$$y = \frac{1}{2}\left(\frac{\pm c}{m} + m\right).$$

उदाहरणम् –

वर्गो नवहतः कश्चिद् दशाढ्यो वा दशोनितः ।
मूलदो जायते तम् गणितज्ञ वद द्रुतम् ॥४२॥

Ex. 42 : "O mathematician, tell me quickly the number whose square, multiplied by 9 the product (either increased or diminished by 10, is a square." ॥ **42** ॥

Statement (*Nyāsa*) : (i). $9x^2 + 10 = y^2$,
(ii). $9x^2 - 10 = y^2$.

Solution :

Case-1: $9x^2 + 10 = y^2$

Here, $a = 3$, $c = 10$, Let m be an optoional number,

Then in accordance with the rule stated in verse 83,

$x = \frac{1}{2a}\left(\frac{\pm c}{m} - m\right) \qquad y = \frac{1}{2}\left(\frac{\pm c}{m} + m\right).$

When $m = 1$,

$x = \frac{1}{2 \cdot 3}\left(\frac{10}{1} - 1\right) = \frac{3}{2}; \qquad y = \frac{1}{2}\left(\frac{10}{1} + 1\right) = \frac{11}{2}$

when $m = 2$,

$x = \frac{1}{2 \cdot 3}\left(\frac{10}{2} - 2\right) = \frac{1}{2}; \qquad y = \frac{1}{2}\left(\frac{10}{2} + 2\right) = \frac{7}{2}$

when $m = 5$,

$x = \frac{1}{2 \cdot 3}\left(\frac{10}{5} - 5\right) = -\frac{1}{2}; \qquad y = \frac{1}{2}\left(\frac{10}{5} + 5\right) = \frac{7}{2}$

Case 2 : $9x^2 - 10 = y^2$

Here, $a = 3$, $c = -10$, Let m be an optional number.

when $m = 1$,

$x = \frac{1}{2 \cdot 3}\left(\frac{-10}{1} - 1\right) = -\frac{11}{6}; \qquad y = \frac{1}{2}\left(\frac{-10}{1} + 1\right) = \frac{-9}{2}$

when $m = 2$

$x = \frac{1}{2 \cdot 3}\left(\frac{-10}{2} - 2\right) = \frac{-7}{6}; \qquad y = \frac{1}{2}\left(\frac{-10}{2} + 2\right) = \frac{-3}{2}$

when $m = 5$

$x = \frac{1}{2 \cdot 3}\left(\frac{-10}{5} - 5\right) = -\frac{7}{6}; \qquad y = \frac{1}{2}\left(\frac{-10}{5} + 5\right) = \frac{3}{2}.$

प्रकृतिसमक्षेपविशुद्धावुदाहरणम् –

का कृतिर्दशभिः क्षुण्णा दशाढ्या वा दशोनिता ।
मूलदा जायते विद्वन् तं द्रुतं वद वेत्सि चेत् ॥४३॥

Ex. 43: "O learned , if you know, tell quickly, the number whose square multiplied by 10, (the product) either increased or dimnished by 10 , happens to be a square." ॥ **43** ॥

Statement (*Nyāsa*): $10x^2 \pm 10 = y^2$,

Solution :

Case : (i). $10x^2 + 10 = y^2$

An obvious solution of the equation, $10x^2 - 10 = y^2$ is,

$(x, y) = (1, 0).$

An obvious solution of the equation, $10x^2 - 1 = y^2$ is,

$(x, y) = (1, 3).$

N = 10	1	0	-10
	1	3	-1

By the principle of Composition,

$l = (1 \cdot 3 + 0 \cdot 1) = 3$; $g = (10 \cdot 1 \cdot 1 + 0 \cdot 3) = 10$;

$i = (-10 \times -1) = 10$. Therefore,

$(x, y) = (3, 10)$ is a solution of $10x^2 + 10 = y^2$.

Case : (ii). $10x^2 - 10 = y^2$

When the optional number is taken as $(m) = 3$, in Sritpati's method , solution of $10x^2 + 1 = y^2$ is

$x = \frac{2m}{N - m^2} = \frac{2 \times 3}{10 - 3^2} = 6, \qquad y = \frac{m^2 + N}{N - m^2} = \frac{3^2 + 10}{10 - 9} = 19.$

Combinining this with the previous solution,

N = 10	1	0	-10
	6	19	1

$l = (1 \cdot 19 + 0 \cdot 6) = 19$; $g = (10 \cdot 1 \cdot 6 + 0 \cdot 19) = 60$;

$i = (-10 \times 1) = -10$. Therefore,

$(x, y) = (19, -10)$ is a solution of $10x^2 - 10 = y^2$.

By subtractive composition also, the same roots are obtained.

अपि च –

क्षयगैकादशघ्नः को वर्गः षष्टिसमन्वितः ।
मूलदो जायते तं मे वद कोविद सत्वरम् [3] ॥४४॥

Ex. 44 : "O learned, tell me quickly, whose square multiplied by -11, (the product) added to 60, happens to be a square." ॥ **44** ॥

Statement (*Nyāsa*): $-11x^2 + 60 = y^2$ ....(1)

Solution :

After finding by whatever means the roots for the given additive, an infinity of them is afterwards deducible by the Princple of composition (*Bhāvanā*) with additive unity and its corresponding roots, which is stated in the next verse. Therefore, we have to find one set of roots of the given equation, and that with interpolator unity.

An obvious solution of (1) is $(x, y) = (1, 7)$.

Now consider the equation $-11x^2 + 12 = y^2$ . . . (2)
An obvious solution of (2) is $(x, y) = (1, 1)$.

Then, by the Principle of Composition,

N = -11	1	1	12
	1	1	12

$l = 1 \cdot 1 + 1 \cdot 1$; $g = (-11 \cdot 1 \cdot 1 + 1 \cdot 1)$; $i = 12 \cdot 12$

$\therefore$ $(x, y) = (2, -10)$ is a set of roots of the equation, $-11x^2 + 144 = y^2$.

[3] . In place of this example, the *Gaṇitakaumudī* gives another similar example. See *Gaṇitakaumudī* Part II, Ex.8 pp.242-243.

So, by the rule stated in verses 74(ii)-75(i), A rational solution of $-11x^2 + 1 = y^2$ is

$$(x, y) = \left(\frac{2}{12} = \frac{1}{6}, \quad \frac{-10}{12} = \frac{-5}{6}\right).$$

Alternatively, by Sritpati's method a rational solution of $-11x^2 + 1 = y^2$ when the optional number taken is $(m) = 1$, is

$$x = \frac{2m}{m^2 - N} = \frac{2 \times 1}{1^2 - (-11)} = \frac{1}{6}, \quad y = \frac{m^2 + N}{m^2 \sim N} = \frac{1 - 11}{12} = \frac{-5}{6}.$$

"Thus , by virtue of the infinite variety of the optional values as well as of the infinitely repeated application of the Principle of Additive and Subtractive Compositions, an infinite number of roots may be found."[4]

3.15 : Alternative rule for Solution of $Nx^2 \pm k = y^2$:

तथा हि –

प्रक्षेपेषु बहुषु (वा) शुद्धिसु वा निजधिया पदे ज्ञेये ।
रूपक्षेपाय तयोर्भावनयाऽनन्तमूलानि ॥८४॥

"When the additive or subtractive is greater than unity, two roots should be determined by one's own intelligence. Then, by combining them with the roots for the additive unity, an infinite number of roots can be obtained."[5] ॥ **84** ॥

[4] . HHM. II. p. 157.

[5] . Cf. (i) BBi. R. ॥८१॥ 92॥ ; (ii)GK.Ch.x. R.15-16. ; (iii). Also see, Br.Sp.Si. xviii-66. HHM, II, p. 174.

यस्य न बुद्धिः स्वांते न गणितलेशोऽपितस्य स्यात् ।
तस्मान्निजया बुद्ध्या समूहयमखिलंतु गणितमिदम् ॥८५॥

"Persons having no intelligence, have no knowledge of mathematics. They should be told the whole of mathematics by one's own intelligence."

उदाहरणम् –

कस्त्रयोदश (सं) निघ्नो वर्गः सप्तदशाधिकः ।
वर्जितो वा पृथङ्मूलप्रदः स्याद् वद मे द्रुतम् ॥४५॥

Ex. 45 : "Tell me that square which being multiplied by 13 and then increased or diminished by 17 becomes capable of yeilding a square-root."[6] ॥ **45** ॥

That is, solve $13x^2 \pm 17 = y^2$

Solution : Ex. (i) $13x^2 + 17 = y^2$

Here, the multiplier = 13 and interpolator = 17.
Now the roots for the interpolator 3 are (1, 4). And for the interpolator 51, the roots are (1, 8). For the composition of these with the previous roots (1, 4) the statement willbe

$$N = 13 \quad l = 1 \quad g = 8 \quad i = 51$$
$$l = 1 \quad g = 4 \quad i = 3$$
$$(1 \times 4 + 8 \times 1 = 12,\ 13 \times 1 \times 1 + 8 \times 4 = 45)$$

So, by the Addition Lemma, we get the roots corresponding to the interpolator 153 as (12, 45).

[6] Cf. GK. X. Ex.9. HHM, II, p. 174-175.

Then in accordance with the rule stated in verse 74, take the optional number to be 3, so that the interpolator may be reduced to 17. For $3^2 = 9$ and $(153/9) = 17$. Therefore, dividing the roots just obtained by the optional number3, we get the required roots $\left(\frac{12}{3} = 4,\ \frac{45}{3} = 15\right)$.
Applying the Subtraction Lemma

$$(8 \times 1 - 1 \times 4 = 4,\ 8 \times 4 - 13 \times 1 \times 1 = 19)$$

and proceeding similarly we get the roots for the interpolator 17 as $(4/3,\ 19/3)$.

Ex. (ii) $13x^2 - 17 = y^2$

Proceeding as before we get (by the Addition Lemma)

N	*l*	*g*	*i*
13	4	15	17
	5	18	-1

$x = 4 \times 18 + 5 \times 15 = 147$
$y = 18 \times 15 + 13 \times 4 \times 5 = 530\ ;\ k = 17 \times -1 = -17$
the roots (147, 530).

And (by the Subtraction Lemma),

N	*l*	*g*	*i*
13	4	15	17
	5	18	-1

$x = 5 \times 15 - 4 \times 18 = 3$
$y = 18 \times 15 - 13 \times 4 \times 5 = 10\ ;\ k = 17 \times -1 = -17$
we get the roots (3, 10).

APPROXIMATE VALUE OF A QUDRATIC SURD

Techniques for finding approximate value of a quadratic surd were known to Indians[1] centuries before Christ. Baudhāyana gave a rule[2] which implies,

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3\cdot 4} - \frac{1}{3\cdot 4\cdot 34}.$$

The same rule is found in the *Śulba Sūtras* of Āpastamba and Kātyāyana.[3] The Berlin Museum Greek Papayrus (A.D.2nd cent) contains[4],

$$\sqrt{164} = 12 + \frac{2}{3} + \frac{1}{15} + \frac{1}{26} + \frac{1}{32}.$$

The approximation,

$$\sqrt{a^2 + r} = a + \frac{r}{2a} - \frac{(r/2a)^2}{2\{a+(r/2a)\}}$$

is found in the Bakṣālī Manuscript.[5] The Jains approximated π by $\sqrt{10}$. Thus approximate values for $\sqrt{10}$ have been given in ancient and medieval jaina works

1. Channabasappa, M. N. On the Squaree-Root Formula in the *Bakhshālī Manuscript, Indian Journal of History of Science (IJHS) Vo.* 11. (1976) pp.112-124, see p. 113.
2. प्रमाणं तृतीयेन वर्धयेत् तच्च चतुर्थेनात्मचतुस्त्रिंशोनेन ।– B.Sl. I-61. Cf. Gupta, R.C. : Vedic Mathematics from the *Śulba Sūtras, IndiaJournal of Mathematics Education* Vol. No.9 No.2, July 1989, pp.2-9. See p.5.
3. Gupta, R. C. : Baudh āyana's Value of $\sqrt{2}$, *The Mathematics Education* vol. VI, No. 3, Sept.1972, Sec. B, pp. 77-79, See p.77. fn.2.
4. Gupta, R. C. : Square root of 164 in the Berlin Papyrus 11529 . *Gaṇita Bhāratī* Vol. No.2, No. 1-2 (1980) pp29-31, See, p. 77.
5. Takao Hyashi : *Bakhshālī Manuscript,* pp 100, and 430-431.

such as the *Anuyogadvāra Sūtra, the Tiloyapaṇṇati and the Tiloya Sāra.* The Jains were able to obtain $\sqrt{10}$ correct to 22 decimal places.[6]

Several other Methods of approximating quadratic surds were prevalent [7]during ancient and medieval periods in cultural areas, as well.

Srīdhara gave a rule[8], according to which

$$\sqrt{N} = \frac{(\sqrt{NA^2})}{A};$$

where, 'N' is the given non-square number, and 'A' is an optionally assumed big number.

The same rule has also been given by Srīpati[9], Bhāskara II [10]and Nārāyaṇa.[11]

6. Gupta, R. C. : Circumference Of The Jambudvīpa In Jain Cosmography, *IJHS,* Vol. 10, No.1, 1975, pp.38-46. See, p .39.
7. Gupta, R. C. : On some ancient and medieval methods of Approximating Quadratic Surds, *Gaṇita Bhāratī* Vol. No.7, No. 1-4 (1985), pp 13-22.
8. *Pāṭīgaṇita,* Ed. by K.S. Shukla with English transalation, p. 91. v R.118.
9. Si.Se. XIII-36;
10. L(ASS)- Vs. 140
11. GK. Part, II, p. 33, Ch.IV. R. 30(ii)-31(i).

अत्र (अ) मूलराशेरासन्नमूलानयने सूत्रम् –

मूलं ग्राह्यं यस्य च रूपक्षेपजे पदे तत्र ।
ज्येष्ठं ह्रस्वपदेन च समुद्धरेन्मूलमासन्नम् ॥८६॥

3.16 : Rule (technique) to find the approximate value of the square-root of a non-square number :

"Obtain the roots (of a square-nature) having unity as the additive and the number whose square-root is to be determined (as the multiplier). Then the greater root divided by the lesser root will be the approximate value of the square-root."[12] ॥ **86** ॥

That is to say, to find the approximate value of the surd $\sqrt{N}$ we shall have to solve the quadratic indeterminate equation :

$$Nx^2 + 1 = y^2 \quad \ldots\ldots\ldots(1)$$

If $x = a,\ y = b$ be a solution of this equation, then,

$\sqrt{N} = \frac{b}{a}$, approximately.

As we know, by the Principle of Composition [vs.71-74(i)] we can derive an infinite number of roots of (1) and thus can derive an infinite number of approximae values of $\sqrt{N}$. Each application of Brahamagupta's Lemma or Additive Composition (*Samāsa-bhāvanā*) will give to $\sqrt{N}$ a value nearer to it. Another intersting aspect of the method is that at any stage of approximation, we know the upper limit and the lower limit between which the error in

12 . Cf. GK. Ch.X. R.17. [GK, Part II, p. 244]

taking the approximate value in place of the exact value, lie.

With the notations of the previous rule [Rationale of Recursive Charcter, in Cyclic method], as ew know, text books on algebra[13] show that the error committed in taking $\frac{p_n}{q_n}$ as the value of the continued fraction i.e., ($\sqrt{N}$ expressed as a continued fraction) is less than $\frac{1}{q_n q_{n-1}}$ and greater than $\frac{a_{n+2}}{q_n q_{n+2}}$.

In fact as Graver, R. has shown[14], the eroor in taking $\frac{p_n}{q_n}$ $\left(or\ \frac{p_{2n}}{q_{2n}}\right)$ as the value of $\sqrt{N}$ is greater than $\frac{1}{2p_n q_n}$ $\left(or\ \frac{1}{2p_{2n}q_{2n}}\right)$ but less than $\frac{1}{2{q_n}^2\sqrt{N}}$ $\left(or\ \frac{1}{2{q_{2n}}^2\ \sqrt{N}}\right)$ according as c is even (or odd).

Thus if $\frac{b}{a}$ is taken as an approximate value of $\sqrt{N}$, where $x = a,\ y = b$ is a solution of the equation, $Nx^2 + 1 = y^2$ then the error in this approximation[15] is greater than $\frac{1}{2ab}$ and less than $\frac{1}{2a^2\sqrt{N}}$.

13 . Barnard, S.and Child, J. M. : *Higher Algebra,* Macmillan Col. Ltd., London, 1952, p. 400.

14 . Graver, Raymond : Concerning Two Square Root Methods, *Bulletin of the Calcutta Mathematical Society,* (*BCMS*), 24(2), 1932, 99-102.

15. For details, See Singh, Paramanand, Nārāyaṇa's Method for evaluating quadratic surds and the regular continued fraction expansionss of the surds , ME, Vol. XVIII, No.32, 63-65.

Nārāyaṇa finds approximations to $\sqrt{10}$ and $\sqrt{(1/5)}$ in the following example, to illustrate his method.

उदाहरणम् –

दशानामपि रूपाणां पञ्चमांशस्य वा वद ।
आसन्नमूलं जानासि यदीमां प्रकृतिक्रियाम् ॥४६॥

Ex. 46 : O friend, if you know the process of *varga-prakṛti*, tell the approximate root of 10 or $1/5$." ॥ **46** ॥

Solution : Case-1 : Consider, $\mathbf{10x^2 + 1 = y^2}$ ......$\mathbf{(1)}$

In accordance with the rule stated in verse [75(ii)-76(i), i.e., Sritpati's Rational Solution], taking the optional number as 3, we find that :

$x = \frac{2m}{m^2 \sim N} = \left(\frac{2\times 3}{3^2 \sim 10}\right) = 6$; $y = \frac{m^2+N}{m^2 \sim N} = \frac{3^2+10}{3^2 \sim 10} = 19$

alternatively, $y = \sqrt{Nx^2 + 1} = \sqrt{\{10 \cdot 6^2 + 1\}} = 19.$

Thus, $(x,\ y) = (6,\ 19)$ is a solutin of (1).

So, $\frac{19}{6}$ will be an approximate value of $\sqrt{10}$.

Now, by the Additive Composition,

N	*l*	*g*	*i*
10	6	19	1
	6	19	1

$x = 19 + 6 \times 19 = 228$; $y = 10 \times 6^2 + 19 \times 19 = 721$

$(x,\ y) = (228,\ 721)$ is another solution of (1).

So, $\frac{721}{228}$ is another approximate value of $\sqrt{10}$.

Now, by the Additive Composition, from the above two solutions, we find that,

N	*l*	*g*	*i*
10	6	19	1
	228	721	1

$x = 6 \times 721 + 228 \times 19 = 8658$;

$y = 10 \times 6 \times 226 + 19 \times 721$
$= 13680 + 13699 = 27379$

$(x,\ y) = (8658,\ 27479)$ is another solution of (1).

Therefore, $\frac{27379}{8658}$ is another value of $\sqrt{10}$ and so on.

Also each subsequent value will be nearer to $\sqrt{10}$ in comparasion to its preceding approximation. The example is very important since $\sqrt{10}$ was widely used as the value of π in ancient world.

Case-2 : Consider, $\mathbf{\left(\frac{1}{5}\right)x^2 + 1 = y^2}$ ......$\mathbf{(2)}$

Taking the optional number as $\frac{1}{2}$, we find that :

$x = \frac{2m}{m^2 \sim N} = \left\{\frac{2(1/2)}{(1/2)^2 \sim (1/5)}\right\} = \frac{1}{(1/20)} = 20$;

$y = \frac{m^2+N}{m^2 \sim N} = \frac{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{5}\right)}{\left(\frac{1}{2}\right)^2 \sim \frac{1}{5}} = \frac{(9/20)}{(1/20)} = 9$

Thus, $(x,\ y) = (20,\ 9)$ is a solutin of (2).

So, $\frac{9}{20}$ will be an approximate value of $\sqrt{(1/5)}$.

Now, by the Additive Composition,

N	*l*	*g*	*i*
$(1/5)$	20	9	1
	20	9	1

$x = 2 \times 20 \times 9 = 360$; $y = (1/5) \times 20^2 + 9^2 = 161$

$(x,\ y) = (360,\ 161)$ as the next solution of(2).

Again , by the Additive Composition, from the above two solutions, we find that,

N	l	g	i
$(1/5)$	20	9	1
	360	161	1

$x = 20 \times 161 + 360 \times 9 \quad = 3220 + 3240 = 6460$;

$y = (1/5) \times 20 \times 360 + 161 \times 9$

$= 1440 + 1449 = 2289.$

$(x,\ y) = (6460,\ 2289)$ is another solution of (2).

Therefore approximate values of $\sqrt{\left(\frac{1}{5}\right)}$ are

$\frac{9}{20}, \frac{161}{360}, \frac{2889}{6460}$, ... and so on.

इति सकलकलानिधिनरसिंहनन्दन–गणितविद्याचतुरानन–
नारायणपण्डितविरचिते बीजगणितावतंसे वर्गप्रकृतिः समाप्ता ।

"Thus ends the "Square-Nature" (वर्गप्रकृतिः) in the *Bījagaṇitāvataṃsa* ("the crown of algebra") composed by Nārāyaṇa Paṇḍita, the Lord Brahma for the lore of (गणित) mathematics, the son of Narasiṃha the abode of all arts."

[समाप्ता बीजक्रिया]

"Thus ends the algebraic operations."

(Part-II)

बीजम्

यस्मादेतत्सकलं विश्वमनंतं प्रजायते व्यक्तम् ।
अव्यक्तादपि बीजाच्छिवं च गणितं च तं नौमि ॥१॥

"As out of Him is derived this entire universe, visible and endless, so out of algebra follows the whole of arithmetic with its endless varieties (of rules). Therefore, I always make obeisance to Siva and also to *(avyakta-) gaṇita* (algebra). ॥ **1** ॥

अव्यक्तसमीकरणं वर्णसमत्वं च मध्यमाहरणम् ।
भावितसम्त्वमस्मिन् बीजानि वदन्ति चत्वारि ॥२॥

"As the equations are classified into four varieties :

(i) Linear equations in one unknown (*avyakta-samīkaraṇa*).

(ii) Linear equations in more than one unknown (*varṇa- samatva or vaṃa-samīkaraṇa*)

(iii) Elimination of the middle term (*madhyamāharaṇa)*, or the quadratic equation.

(iv) Equations involving the product of different unknowns (*bhāvita-samatva*)."

analysis is stated to be of four kinds." ॥ **2** ॥

तत्रादौ तावदव्यक्तसाम्ये करणसूत्रमार्यापञ्चकम् –

यावत्तावच्चिह्नितमेकं वा बहुमितं तु परिकल्प्य ।
रूपाढ्यं वा रूपोनितमथवाऽव्यक्तमानमिति ॥३॥
माने तस्मिन्नेवोद्देशालापवत् समाचरेत् कर्म ।
फलसिद्ध्यै द्वौ पक्षौ तुल्यौ कार्यौ प्रयत्नेन ॥४॥

At the out set, Rules for the formation (and solution) of (linear) equation (in one unknown) [in 5 verses of *āryā metre*]:

"Let *yāvat-tāvat* with one, (two,) or any number as a coefficient of it, with or without an absolute term, be assumed as the value of the unknown quantity.

Then, on the value thus assumed, by performing the operations (such as addition, subtraction, multiplication, division, etc.,) in accordance with the statement of the problem, the two equal sides (of an equation) should be very carefully built, to get the desired result."[1] ॥ **3-4** ॥

बीजचतुष्टयमाद्याः प्राहुः । तेषु प्रथमे तावदव्यक्तसाम्ये यत्रोदाहरणे योऽज्ञातो राशिस्तन्मानं यावदेकं द्व्यादि वा सरूपमरूपं वा रूपैरूनं युतं वा प्रकल्प्य तस्य राशेर्वासनानुसारेणोद्देशकालाप–वद्योगवियोगगुणनभजनत्रैराशिकपञ्चराशिकश्रेढिक्षेत्रखातादिक्रियया द्वौ पक्षौ समौ कार्यौ । यस्मिन्नालापे पक्षयोः समत्वं न विद्यते तदेकपक्षः केनचित्संगुणितो भक्तो युतो वर्जितो वा निजबुद्ध्या पक्षौ समौ कार्यौ।

"Of these (four classes of equations), the linear equation in one unknown (will be treated) first. In a problem (proposed), the value of the quantity which is unknown is assumed to be *yāvat,* one, two or any multiple of it, with or without an absolute term, which again may be additive or subtractive. Then on the value thus assumed optionally should be performed, in accordance with the statement of the problem, the operations such as addition, subtraction, multiplication, division, rule of three, summation, plane figures, excavations, etc. And thus the two sides must be made equal. If the equality of the two sides is not explicitly stated, then one side should be multiplied, divided, increased or decreased by one's own intelligence (according to the problem) and thus the two sides must be made equal."

[1]. (i). Br.Sp.Si.xviii. 43. Pr. Comm.. ; (ii). NBi. II. R. 3-4.(Gloss) Cf. HHM. II. p.29.

एकस्मादव्यक्तं विशोधयेदन्यतस्तु रूपाणि ।
शेषेणाव्यक्तेन च समुद्धरेद्रूपशेषमिह ॥५॥
अव्यक्तस्य च राशेर्मानं व्यक्तं प्रजायते नूनम् ।

" From one side clear off the unknown and from the other the known quantities ; then divide the residual known by the residual coefficient of the unknown. Thus will certainly become known the value of the unknown."[2] ॥ **5-6(i)** ॥

आज्ञातेषु बहुषु वा यावत्तावद् द्विकादिसंगुणितम् ॥६॥
भक्तं रूपैर्युक्तं विवर्जितं वा प्रकल्पयेदेवम् ।
निजबुद्ध्या बिज्ञेयं क्वचिदव्यक्तस्य च मानम् ॥७॥

[2]. (i). Ā. ii. 30. ; (ii). Br.Sp.Si.xviii. 43. ; (iii). BBi. R. 101. Cf. HHM. II. p. 41.

"In case of two or more unknowns, multiplied by 2 etc. (i.e., by arbitrary known numbers), or divided, increased or decreased by them, or in some cases, according to one's own sagacity (simply) any known values may be assumed for the other unknowns. [Knowing these the rest is an equation in one unknown].[3] ॥ **6(ii) – 7** ॥

उदाहरणानि –

समानमौल्या वणिजोऽष्टघोटा
एकस्य रूपाणि शतानि षट् च ।
ऋणे शतेऽन्यस्य च वाजिनोऽर्क–
मिताः समौ तौ च किमश्वमूल्यम् ॥१॥

(अपूर्णम्)

Ex. 1 : "One merchant has 8 horses of similar value, and six hundred coins. Another has 12 horses and a debt of (one) hundred coins. Both of them are of equal worth. What is the price of a horse?" ॥ 1 ॥

Solution : Here the statement for equi-clearance is :

$$8x + 600 = 12x - 100.$$

By the rule steted above in verses 5-6(i), unknown of the firstside being subtracted from the unknown on the other side, the remainder is $4x$. The absolute term on the second being subtracted from the first side, the remainder is 700. The residual known number 700 being divided by the coefficient of the residual unknown $4x$, the quotient is recognised tobe the value of x, namely 175.

INCOMPLETE

[3]. BBi. ॥८१॥ 102 ॥ ; HHM. II. pp. 126 - 127.

Appendix - I

Residual Pulverisers and Conjunct Pulverisers (*sāgra-kuṭṭaka and saṁśliṣṭa- kuṭṭaka*) from *Gaṇitakaumudī*

2.14.1: GENERAL PROBLEM OF REMAINDERS **:** The general problem of remainders , *viz.*, to find a number N which being severally divided by a_1, a_2, a_3, ……, a_n leaves as remainders $r_1, r_2, r_3, \ldots\ldots, r_n$ respectively, sybolically,:

$$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \ldots = a_nx_n + r_n$$

is one of the types of problems of simultaneous indeterminate equations of the first degree. Here a_1, a_2, a_3, ……, a_n are called 'divisors', (*bhāgahāra, bhājaka and cheda* etc.) and $(r_1, r_2, r_3, \ldots\ldots r_n)$ as 'remainders' (*agra* and *śeṣa* etc.).

सूत्रम् ।

आद्यो हारो हारं परो विभाज्यं प्रकल्प्य पूर्वाग्रम् ।
त्यक्त्वा पराग्रतस्तच्छेषं क्षेपं च तल्लब्ध्या ॥ ३२ ॥
गुणितः प्रथमो हारः साग्रोऽग्रं भाज्यताडितस्तु हरः ।
सोऽस्याद्यः स्यादेवं तदग्रमपरोऽपि राशिः स्यात् ॥३३॥

R. 32-33 : "Supposoing the first divisor to be the divisor, the next divisor to be the dividend and the difference of the first remainder subtracted from other remainder to be the additive, find the quotient. Multiply it by the first divisor and add (the product) to the first

remainder. (After that,) suppose (the sum) to be the remainder (and) the product of the first divisor (and) the next divisor, the divisor, (Now), these become the first (remainder and the first divisor) and, also , that next to it, the other (and so on) and thus finally, the remainder happens to be the (desired) number."||32-33||

Explanation and Rationale : The rule gives a method to find N, which when separately divided by $a_1, a_2, a_3, \ldots\ldots, a_n$ leaves $r_1, r_2, r_3, \ldots\ldots, r_n$ as remainders in order, i.e., to find N such that,

$$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \ldots = a_n + r_n \quad \ldots(1)$$

For definiteness, let us consider,

$$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 \qquad \ldots(2)$$

According to the rule, let us first solve

$$N = a_1x_1 + r_1 = a_2x_2 + r_2 \qquad \ldots\ldots\ldots(3)$$

i.e., $a_2x_2 + (r_2 - r_1) = a_1x_1$

or $$\frac{\{a_2x_2 + (r_2 - r_1)\}}{a_1} = x_1$$

Let $x_1 = m$, $x_2 = b_2$ be the minimum solution of this equation. Then, the least value of N satisfying (3) will be $a_1m + r_1$. Hence the general value of N satisfying (3) will be

$$N = a_1(m + a_2u) + r_1,$$
$$= a_1a_2u + a_1m + r_1.$$

where u is any integer.

Now, consider,

$$N = a_1a_2u + a_1m + r_1 = a_3x_3 + r_3. \qquad \ldots\ldots(4)$$

i.e., $$\frac{a_3x_3 + \{r_3 - (a_1m + r_1)\}}{a_1a_2} = u$$

Let $u = n$, $x_3 = b_3$ be the minimum solution of this equation. Then, the least value of N satisfying (4) will be $(a_1a_2n + a_1m + r_1)$. This value of N will also satisfy (3), as this value has been obtained from the values of N satisfying (3). Therefore, the general value of N satisfying (3) and (4) simultaneously, i.e., satisfying (2) will be given by

$$\mathrm{N} = a_1a_2a_3v + a_1a_2n + a_1m + r_1 \quad \ldots\ldots(5)$$

Proceeding in this way, we will obtain the values of N satisfying (1).

The process of solution of equation (1) was known to Āryabhaṭa I (499 CE)[1]. Bhāskara I (629 CE) has interpreted[2] Āryabhaṭa I's rule as a method for the solution of this equation. Brahamagupta (628 CE) has[3] also given the rule. Bhāskara II (1250 CE) has given two methods for the solution fo this equation. one of this method is the same as that given[4] by Āryabhaṭa I. One of the Bhāskara I's problem[5] was later on dealt with by[6] Ibn-al-Hatiam (c.1000 CE) and Leonardo Fibonacci of Pisa(c. 1202 CE).

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[1] . Cf. Ā-ii-32-33, pp.74-77;.

[2] . Cf. Ā-ii-32-33, p.77; ABh, p.

[3] . Cf. BrSpSi.xviii. 3-5.

[4] Cf. HHM, II. pp. 133-135

[5] द्व्याद्यैः षट् पर्यन्तैरेकाग्रः योऽवशिष्यते राशिः ।
सप्तिभिरेव स शुद्धो वदशीघ्रं को भवेद् गणक ॥ 4 ॥

-ABh. p. 134. and p.

[6] . Cf. HHM, II. p. 133.

(10). उदाहरणम् ।

द्व्यग्रस्त्रिहृतस्त्र्यग्रश्चतुराप्तः पञ्चहृच्चतुष्काग्रः ।
पञ्चाग्रः षड्भक्तो यस्तं कथयाशु मे गणक ॥३०॥

EX.10: " O Mathematician, tell me quickly (the numbers) which when divided by 3, 4, 5 and 6 leave remainders 2, 3, 4 (and) 5, in order." ‖ 30 ‖

statement (Nyāsa) :

remainder	2	3	4	5
divisor	3	4	5	6

Solution : The equqtion to be formed will be

$$N = 3x + 2 = 4y + 3 = 5z + 4 = 6t + 5.$$

Least value of x satisfying the equation,

$$3x + 2 = 4y + 3 \quad \text{or} \quad \frac{(4y+1)}{3} = x \quad \text{will be}$$

```
3 ) 4 ( 1
    3
    1
```

1	$1\times 1 + 0 = 1$
1	1
0	-

will be $x = 4 - 1 = 3$.
So, the general value will be : $x = 3 + 4m$
where m is any integer. Substituting this value of x in $(3x + 2)$, we get $\{3(3 + 4m) + 2\} = 12m + 11$.

Now, consider the equation,

$$12m + 11 = 5z + 4 \quad \text{or} \quad \frac{5z-7}{12} = m$$

Now find the least value of m satisfying the equation,


```
12 ) 5 ( 0
     0
     5 ) 12 ( 2
         10
          2 ) 5 ( 2
              4
              1
```

0			$0\times 35 + 14 = 14$
2		$2\times 14 + 7 = 35$	35
2	$2\times 7 + 0 = 14$	14	-
7	7	-	-
0	-	-	-

$(35 = 12 \times 2 + 11)$; $(14 = 5 \times 2 + 4)$

Least value of m satisfying the equation, will be 4, and so its general value will be $5n + 4$, where n is any integer. Substituting this value of m in $(12m + 11)$.

weget. $\{12(5n + 4) + 11\} = 60n + 59$

Finally, consider the equation,

$$60n + 59 = 6t + 5 \quad \text{or} \quad 6t - 54 = 60n$$

Dividing by 6, the H.C.F. of the dividend, divisor, and interpolator weget, $t - 9 = 10n$ or $\frac{t-9}{10} = n$

1	0	)	1	(	0
			0		
			1		

0	$0 \times 9 + 0 = 0$
9	9
0	-

Least value of n satisying this equation will be 0, so, its generalvalue will be $(0 + p)$, where p is any integer.
Substituting this value of n in $(60n + 59)$ we get.

$N = 60p + 59$, where p is any integer. as general solution of (1), its least solution being 59.

Examples like (1) are now known as 'the Chinese problems on remainders'. The first such problem occurs in a work of Sun Tsu, who could get only one solution of it.

Even by the time of 6th and 7th centuries, three successive Chinese mathematicians obtained only 3 successive solutions. According to Mikami the Chinese interest in indeterminate anaalysis grew after their contact with Hindu culture and under the influence of the latter.[7]

(11). अपि च ।

को राशिश्चतुरूनः सप्तविभक्तस्तु शुद्धिमुपयाति ।
सप्तयुतो नवभक्तस्त्र्यूनो दशभाजितः कः स्यात् ॥ ३१ ॥

EX.11: "What are the numbers which when lessened by 4 and 3 or increased by 7, are exactly divided by 7, 10and 9, in order." ॥ 31 ॥

Statement (Nyāsa) :

remainder	4	-7	3
divisor	7	9	10

Solution : The equqtion to be formed will be

$$N = 7x + 4 = 9y - 7 = 10z + 3.$$

So, the equation to be solved at the outset is,

$$9y - 11 = 7x \quad \text{or} \quad \frac{9y-11}{7} = x$$

7) 9 (1
$\underline{7}$
2) 7 (3
$\underline{6}$
1

1	1	1× 33+11=44
3	3× 11+0=33	33
11	11	-
0	-	-

$44 = 9 \times 4 + 8.$

[7] . Shukla, K. S. : Hindu mathematics in the 7th century as found in Bhaskara- I's Commentary on Aryabhatiya, *Ganita,* Vol.23, (June 1972), No.3 58-59. Cf. Paramananad Singh : English Translation with notes of Ganita kaumudi , *Ganita Bharati ,* Vol.22 Nos. 1-4 (2000), 19-85 see p. 45

Since the interpolator is nagative, the least value of x in this case is : $9 - 8 = 1$. So, the general value of x is, $x = 1 + 9m$, where m is any integer.

Substituting this value of x in $(7x + 4)$, we get,

$$7(9m + 1) + 4 = 63m + 11.$$

Now, consider the equation,

$63m + 11 = 10z + 3$ or $10z - 8 = 63m$

6 3) 1 0 (0
0
1 0) 6 3 (6
6 0
3) 1 0 (3
9
1

0	0	0	0× 152+24=24
6	6	6× 24+8=152	152
3	3× 8+0=244	24	-
8	8	-	-
0	-	-	-

$$24 = 10 \times + 4$$

Least value of m satisfying the equation, will be 4, and so its general value will be $(4 + 10n)$, where n is any integer. Substituting this value of m in $(63m + 11)$.

$\therefore N = 63(10n + 4) + 11$
$= 630n + 263$; where n is any integer.

2.14.2 : CONJUNCT PULVERISER (संश्लिष्टकुट्टक) :

Let the The general problem on remainders be,
$a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \;\dots\; = a_nx_n + r_n \dots (1).$
Then[1],

$$a_1x_1 + r_1 = a_2x_2 + r_2$$
$$a_2x_2 = a_1x_1 + (r_1 - r_2) \dots\dots\dots (i)$$

$$a_1x_1 + r_1 = a_3x_3 + r_3$$
$$a_3x_3 = a_1x_1 + (r_1 - r_3)$$
$$\therefore x_3 = \left\{\frac{a_1}{a_3}x_1 + \frac{1}{a_3}(r_1 - r_2)\right\}$$

$$a_2x_3 = \frac{a_1a_2}{a_3}x_1 + \frac{a_2}{a_3}(r_1 - r_3) \dots\dots\dots (ii)$$

$$a_2x_4 = \frac{a_1a_2}{a_4}x_1 + \frac{a_2}{a_4}(r_1 - r_4) \dots\dots\dots (iii)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Thefore the system of indeterminate equations of the first degree :
$$a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \;\dots\; = a_nx_n + r_n$$
can be put into the form
$$\left(\begin{array}{l} by_1 = a_1x \pm c_1 \\ by_2 = a_2x \pm c_2 \\ by_3 = a_3x \pm c_3 \\ \dots\dots\dots\dots\dots\dots. \end{array}\right) \dots\dots\dots. (1)$$

This is technically called *saṁsliṣta-kuṭṭaka* or the "conjunct pulveriser" (from, *kuṭṭaka* = pulveriser and *saṁsliṣta* = joined together, related).

[1] . HHM. II. p.135. f.n.

On account of its important applications in mathematical astronomy this modified system has received special treatment at the hands of Hindu algebraists from Aryabhata II (950) onwards.

For the solution of the above system of equations Aryabhata II lays down the following rule :

"In the solution of simultaneous indeterminate equations of the first degree with a common divisor, the dividend will be the sum of the multipliers and the interpolator the sum of the interpolators."[2]

A similar rule is given by Bhāskara II. He says[3] :

"If the divisor be the same but the multipliers different then making the sum of the multipliers the dividend and the sum of residues the residue (of a pulveriser), the investigation is carried on according to the foregoing method. This true method of the pulveriser is called *saṁslişta-kuṭṭaka,* the conjuct pulveriser."

Rationale : If the equations (1) are satisfied by some value 'a' of x, then the same value will satisfy the equation
$$b(y_1 + y_2 + \;\dots) = (a_1 + a_2 + \;\dots)x + (c_1 + c_2 + \cdots) \dots (2)$$

Thus, if we can find the general value of x satisfying equation (2), one of these values, at least, will satisfy all the equations of (1).

[2] गुणैक्यं संश्लिष्टे भाज्यः शेषैक्यं भवेत् क्षेपः ॥ MSi. Xviii-48

[3] . एको हरश्चेद्गुणकौ विभिन्नौ तदा गुणैक्यं परिकल्प्य भाज्यम् ।
अग्रैक्यमग्रं कृत उक्तवद्यः संश्लिष्टसंज्ञः स्फुटकुट्टकोऽसौ ॥
– L(ASS). Vs.259; B. Bi.॥६८॥73॥

To illustrate the application of the above Bhāskara II gives the following example :

EX. "What quantity is it, which multiplied by five and divided by sixty-three, gives a residue of seven; and the same multiplied by ten and divided by sixty-three, a remainder of fourteen ? declare the number."[4]

That is, we have to solve the Simultaneous Indeterminate Equations :

$$63y_1 = 5x - 7 \quad \ldots\ldots(i)$$
$$63y_2 = 10x - 14 \quad \ldots\ldots(ii)$$

Solution : Adding up the equations

$$63(y_1 + y_2) = (5 + 10)x - (7 + 14),$$

and dividing by the common factor 3, we get

$$21Y = 5x - 7,$$

where $Y = y_1 + y_2$.

21) 5 (0
0
5) 21 (4
20
1

0	0	0 x 28 + 7 = 7
4	4x7 + 0 = 28	28
7	7	-
0	-	-

7 = 1 x 5 + 2 ; 28 = 1 x 21 + 7

Since the interpolator is negative : $x = 21 - 7 = 14$.
By the method of the pulversier the least positive value of x satisfying this equation is $x = 14$. This value of x is found to satisfy both the equations (i) and (ii).

---

[4] . कः पञ्चनिघ्नो विहृतस्त्रिषष्ट्या
सप्तावशेषोऽथ स एव राशिः ।
दशाहतः स्याद् विहृतस्त्रिषष्ट्या
चतुर्दशाग्रो वद राशिमेनम् ॥६१॥ 74 ॥

-HHM. II. pp. 136- 137.

2.14.3 : GENERALISED CONJUNCT PULVERISER.

(*saṁśliṣṭa- kuṭṭaka*)

A generalised conjunct pulveriser is that in which the divisors as well as multipliers vary. Thus we have

$$b_1y_1 = a_1x \pm c_1$$
$$b_2y_2 = a_2x \pm c_2$$
$$b_3y_3 = a_3x \pm c_3$$
. . . .

Simultaneous indeterminate equations of this type have been treated by Mahāvīra (850) [Cf.GSS vi. 115½, 136½] and Śripatī (1039). [SiSe, xiv, 28.].

In this connection the rule stated by Nārāyaṇa Paṇḍita is as follows:

सूत्रम् ।

भाज्यं गुणकारोऽग्रं क्षेपं हारो हरं प्रकल्प्याथ ।
कुट्टकजो यो गुणकः स निजहराग्रं विधिः प्राग्वत् ॥३४॥

R.34 : "Supposing the multilpier (*guṇaka. or guṇakāra*) to be the dividend (*bhājya*), the remainder (*agra*) to be the interpolator (*kṣepa*), and the divisor (*bhājaka, hara*) to be the divisor, obtain the multilpiers by the method of *kuṭṭaka* (i.e., by the method of solution of indeterminate equations of the first degree). These (multipliers) are the corresponding remainders of their own divisors. (The desired number should be obtained from them) by the method stated earlier (inverse 32-33)." ॥34॥

Explanation and Rationale : Let

$$b_1y_1 = a_1x \pm c_1 \quad ......(1)$$
$$b_2y_2 = a_2x \pm c_2 \quad .....(2)$$
$$b_3y_3 = a_3x \pm c_3 \quad .....(3)$$

then according to the rule, the value of N can be obtained from the equations,

$$N = b_1y_1 + m = b_2y_2 + n = b_3y_3 + p$$

where m, n, p are the least value of N satisfying equations (1), (2), (3) in order, and hence the rule.

Bhāskara II has given rules for the solution of such type of equations.[5] However, some mathematicians doubt the genuineness of the rule[6] given by Bhāskara II while some others take a different view.[7]

(12). उदाहरणम् ।

को राशिर्निधिशैलसायकगुणैर्निघ्नः पृथग् भाजितो
बाणेभेशपुरन्दरैः क्षितिकराग्न्यम्भोधिशेषो भवेत् ।
तं राशिं वद कोविदाशु गणकाहङ्कारशैलस्थली–
वासिप्रोन्मदकुट्टकज्ञकरिणां जेता नृसिंहोऽसि चेत् ॥३२॥

Ex. 12 : "O mathematician, if you are just like a lion residing in mountains, to win the elephants, among the learned mathematicians knowing Indeterminate

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5 . S. K. Ganguly, "Bhaskaracarya and simultaneous indeterminate equations of first degree," BCMS, XVII, 1926, pp.89-98.

6 . A. A. Krishnaswami Ayyangara, "Bhaaskara anda samslishta Kuttaka" JIMS, XVIII, 1929

7 . Ganguly's reply to Ayyangar's criticism see JIMS, XIX, 1931. Cf.HHM, II, p. 139.

Analysis, tell quickly the numbers which when multiplied by 9, 7, 5 and 3 and divided by 5, 8, 11 and 14 leave remainders 1,3, 3, and 4 in order." ॥32॥

Statement:

multiplier	9	7	5	3
divisor	5	8	11	14
remainder	1	2	3	4

Here, the problem is to find the values of N which satisfy all the following equations simultaneously,

$9N - 1 = 5x$; $7N - 2 = 8y$; $5N - 3 = 11z$; $3N - 4 = 14t$

Solving these equations :

(i) $9N - 1 = 5x$

```
5 ) 9 ( 1
    5
    4 ) 5 ( 1
        4
        1
```

1	1	1× 1 + 1=2
1	1× 1+0=1	1
1	1	-
0	-	-

Since the interpolator is nagative, the least value of N in this case is : 5-1=4

(ii) $7N - 2 = 8y$

```
8 ) 7 ( 0
    0
    7 ) 8 ( 1
        7
        1
```

0	0	0× 2+2=2
1	1× 2+0=2	2
2	2	-
0	-	-

The least value of N in this case is : $8 - 2 = 6$.

(iii) $5N - 3 = 11z$

```
1 1 ) 5 ( 0
      0
      5 ) 1 1 ( 2
          1 0
            1
```

0	0	0× 6 + 3 = 3
2	2× 3 + 0 = 6	6
3	3	-
0	-	-

The least value of N in this case is : $11 - 6 = 5$.

(iv) $3N - 4 = 14t$

```
1 4 ) 3 ( 0
      0
      3 ) 1 4 ( 4
          1 2
            2 ) 3 ( 1
                2
                1
```

0	0	0	0× 20+4=4
4	4	4× 4+4=20	20
1	1× 4+0=4	4	-
4	4	-	-
0	-	-	-

$$20 = 14 \times 1 + 6$$

Since in the mutual division the number of quotients is odd, and also the interpolator is negative.

Therefore, the least value of N in this case is : 6.

Now, aaccording to the rule, these (least value of N or multipliers) are the corresponding remainders of their own divisors. That is,

remainder	4	6	5	6
divisor	5	8	11	14

Thus, according to the rule, we have to find the value of N, where N satisfies the equations,

$N = 5x + 4 = 8y + 6 = 11z + 5 = 14t + 6$

(i). $8y + 2 = 5x$ or $\frac{8y+2}{5} = x$

```
5 ) 8 ( 1
    5
    3 ) 5 ( 1
        3
        2 ) 3 ( 1
            2
            1
```

1	1	1	1× 4+2=6
1	1	1× 2+2=4	4
1	1× 2+0=2	2	-
2	2	-	-
0	-	-	-

Since in the mutual division the number of quotients is odd, and the interpolator is positive, the least value of $x = 8 - 6 = 2$. ∴ The general value is $x = 2 + 8m$ where m is an integer. Substituting this general value of x in $(5x + 4)$ and then equating it with $11z + 5$, we get:

$$5(2 + 8m) + 4 = 11z + 5$$

or $$40m + 14 = 11z + 5$$

or $$11z - 9 = 40m.$$

```
4 0 ) 1 1 ( 0
        0
      1 1 ) 4 0 ( 3
            3 3
              7 ) 1 1 ( 1
                    7
                    4 ) 7 ( 1
                        4
                        3 ) 4 ( 1
                            3
                            1
```

0	0	0	0	0	0× 99+27=27
3	3	3	3	3× 27+18=99	99
1	1	1	1× 18+9=27	27	-
1	1	1× 9+9=18	18	-	-
1	1× 9+0=9	9	-	-	-
9	9	-	-	-	-
0	-	-	-	-	-

$27 = 11 \times 2 + 5;$

Since in the mutual division the number of quotients being 5 is odd, and also the interpolator is negative,

the least value of $m = 5$ ∴ The general value is $m = 5 + 11n$ where n is an integer. Substituting this

general value of m in $(40m + 14)$ and then equating it with $14t + 6$, we get:

$$40(5 + 11n) + 14 = 14t + 6$$
$$440n + 214 = 14t + 6$$
$$14t - 208 = 440n$$

or
$$7t - 104 = 220n.$$

```
220 ) 7 ( 0
      0
      7 ) 220 ( 3
          217
            3 ) 7 ( 2
                6
                1
```

0	0	0	0× 6552+208=208
31	31	31× 208+104=6552	6552
2	2× 104+0=208	208	-
104	104	-	-
0	-	-	-

$$208 = 7 \times 29 + 5$$

Since in the mutual division the number of quotients being 3 is odd, and also the interpolator is negative,
the least value of n= 5 ,
$\therefore$ The general value is n= $5 + 7p$ where p is an integer.

Substituting this general value of n in $(440n + 214)$ we get the desired value of N,

$$\therefore N = 440(7p + 5) + 214$$
$$= 3080p + 2200 + 214$$
$$= 3080p + 2414\ ;$$

where, $p = 0, 1, 2, 3, ..$

2.14.4 : GENERAL PROBLEM ON REMAINDERS (CONTINUED)

सूत्रम् ।

प्राग्वद्राशिः साध्यस्तच्छेषहरौ समीरितहराप्तौ ।
तल्लब्धं प्रथमः स्यादुद्दिष्टहराग्रगो द्वितीयश्च ॥ ३५ ॥
ताभ्यां कुट्टकलब्ध्या राशिहरस्ताडितो निजाग्रयुतः ।
परहरगुणितो हारो मुहुर्विधिश्चैवमन्येषु ॥ ३६ ॥

R.35-36: "Obtain the quantities (i.e., the divisor and the remainder) by the method stated earlier (in verse 32-33). (Divide the quantities by the first divisor) The quotients are the first (divisor) and the first remainder. Desired divisor and the (desired) quotients are the second. Obtain the quotients from these by *kuṭṭaka* (i.e., by the method of Indeterminate Analysis). Multiply the quotient by the (initial) divisor. Add its own (i.e., initial) remainder to the product. The sum is the next remainder. Multiply the (initial divisor) by the other divisor. (The product) is the next divisor. In this way, other (divisor and remainders) should be obtained by repeating the process." ॥ 35-36 ॥

Explanation and Rationale : The above rule is a generalisation of R.32-33 and gives several other solutions of the equation,

$$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \ldots = a_n + r_n \ldots (1)$$

For definiteness, let us consider,

$$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 \qquad \ldots (2)$$

We have seen in R.32-33 that relation (5) of that rule,
i.e., N = $a_1a_2a_3v + a_1a_2n + a_1m + r_1$ ….…. (3)
is a solution of (2), where v is any integer and $x_1 = m$, $x_2 = b_2$ is the minimum solution of $a_1x_1 + r_1 = a_2x_2 + r_2$ and $u = n, x_3 = b_3$ is the ninimum solution of $a_1a_2u + a_1m + r_1 = a_3x_3 + r_3$.

Now, divide $a_1a_2a_3v$ and $(a_1a_2n + a_1m + r_1)$ by a_1. This will give a_2a_3v and $(a_2n + m)$ as quotients.

Consider the equation,

$$a_2a_3v + a_2n + m = a_1x_1 + r_1 \quad \ldots (4)$$

Let $v = f$ and $x_1 = d_1$ be the minimum solution of this equation. According to the rule,

$$N = a_1^2a_2a_3w + a_1a_2a_3f + a_1a_2n + a_1m + r_1 \quad \ldots.(5)$$

where w is any integer.
This value of N will also satisfy (2) as this value has been derived from (3) which satisfies (2).
Again suppose that q_2 is the quotient when $(a_1m + r_1)$ is divided by a_2. Obviously, the remainder will be r_2 in this case as (3) stisfies (2) as well.
That is, $a_1m + r_1 = a_2q_2 + r_2$ ……….. (6)
Now, division of $a_1^2a_2a_3w$ and $a_1a_2a_3f + a_1a_2n + a_1m + r_1$ by a_2 will give the quotients $a_1^2a_3w$ and $(a_1a_3f + a_1n + q_2)$.
Let $w = g$, $x_2 = d_2$ be the minimum solution of

$$a_1^2a_3w + a_1a_3f + a_1n + q_2 = a_2x_2 + r_2 \quad \ldots.. (7)$$

Then,

$$N = a_1^2a_2^2a_3k + a_1^2a_2a_3g + a_1a_2a_3f + a_1a_2n + a_1m + r_1 \text{-(8)}$$

where k is any integer.

This value of N will also satisfy (2) as this value has also been derived from (3) which satisfies (2).
Finally, suppose that q_3 is the quotient when, $(a_1a_2n + a_1m + r_1)$ is divided by a_1.
Obviously, the remainder in this case will be r_3 as (3) satisfies (2) as well. So,

$$(a_1a_2n + a_1m + r_1) = a_3q_3 + r_3 \quad \ldots \ldots (9)$$

Now division of, $a_1^2a_2^2a_3k$ and $(a_1^2a_2a_3g + a_1a_2a_3f + a_1a_2n + a_1m + r_1)$ by a_3 will give the quotients $a_1^2a_2^2k$ and $(a_1^2a_2g + a_1a_2f + q_3)$.
Let $k = h$ and $x_3 = d_3$ be the minimum solution of

$$a_1^2a_2^2k + a_1^2a_2g + a_1a_2f + q_3 = a_3x_3 + r_3 \quad \ldots(10)$$

Then,

$$N = a_1^2a_2^2a_3^3j + a_1^2a_2^2a_3h + a_1^2a_2a_3g + a_1a_2a_3f + a_1a_2n + a_1m + r_1 \quad \ldots(11)$$

where, j is any integer. This value of N will also satisfy (2) as this value has been derivied from (3) which satisfies (2) and so on and hence the rule.

Here, it may be noted that all the solutions [i.e., values of N given by (5) of R.32-33 or (5), (8), and (11) of R.35-36] of the equatio (1) of R.32-33 or, R.35-36 are transferable from one form to the other by simply interchanging the optional numbers, suitably. Thus if we take optional number w in place of v such that $v = a_1w + f$, where f is the number satisfying equation (4) of R.35-36, we get solution (5) of R.35-36 in place of solution (5) of R.32-33 of the equation (2) of R.32-33 (or R.35-36) and so on.

(13). उदाहरणम् ।

एकाग्रस्त्रिहृतः कः स्यात् त्र्यग्रः पञ्चविभाजितः ।
पञ्चाग्रः सप्तभक्तश्च तद्वदेव पृथक् फलम् ॥३३॥

Ex.13 : "Tell the numbers which when separately divided by 3, 5, and 7, leave remainders 1, 3 and 5, (in order)." ॥33॥

Statement (Nyāsa) :

remainder	1	3	5
divisor	3	5	7

That is, to find the value of N such that,

$$N = 3x + 1 = 5y + 3 = 7z + 5 \quad \ldots(1)$$
$$= a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3$$

Solution : We have seen in R.32-33 that

$$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3$$

has the general solution

$$\mathrm{N} = a_1a_2a_3v + a_1a_2n + a_1m + r_3 \quad \ldots..(2)$$

where v is any integer and $x_1 = m,\ x_2 = b_2$ is the minimum solution of $a_1x_1 + r_1 = a_2x_2 + r_2$ and $u = n, x_3 = b_3$ is the ninimum solution of $a_1a_2u + a_1m + r_1 = a_3x_3 + r_3$.

So, first we have to find the values of m and n, that is, minimum solution of

$$3x + 1 = 5y + 3 \text{ or } \frac{5y+2}{3} = x \quad \ldots\ldots(3)$$

$$\begin{array}{l} 3\,)\,5\,(\,1 \\ \quad \underline{3} \\ \quad 2\,)\,3\,(\,1 \\ \qquad \underline{2} \\ \qquad 1 \end{array}$$

1	1	1× 2 + 2 = 4
1	1 × 2 + 0 = 2	2
2	2	-
0	-	-

$$4 = 5 \times 0 + 4; \quad 2 = 3 \times 0 + 2$$

Therefore minimum solution of the equation (3) or the minimum values of x and y are : $x = 4$ and $y = 2$.

The general solution of (3) is : $x = 4 + 5u$; $y = 2 + 3u$ where u is any integer.

Substituting the value of x and y in (3) and equating it with $7z + 5$ we get, $3(4 + 5u) + 1 = 7z + 5$

$$\therefore\ 15u + 13 = 7z + 5 \text{ or } \frac{7z-8}{15} = u \quad \ldots..(4).$$

0	0	0 × 16 + 8 = 8
2	2 × 8 + 0 = 16	16
8	8	-
0	-	-

$$\begin{array}{l} 15\,)\,7\,(\,0 \\ \quad \underline{0} \\ \quad 7\,)\,15\,(\,2 \\ \qquad \underline{14} \\ \qquad 1 \end{array}$$

$8 = 7 \times 1 + 1$; $16 = 15 \times 1 + 1$

Since the interpolator is negative, The minimum solution of (4) is $(u,\ z) = (7 - 1,\ 15 - 1)$ i.e., $u = 6, z = 14$.

Therefore the general solution: $u = 6 + 7v$; $z = 14 + 15v$; where v is any integer.

So, $x_1 = m = 4$ and $u = n = 6$. Substituting these values in (2), we get,

$$N = 3 \cdot 5 \cdot 7 \cdot v + 3 \cdot 5 \cdot 6 + 3 \cdot 4 + 1$$
$$= 105v + 103 \quad \ldots \quad \ldots.. \quad \ldots\ldots(5)$$

As stated in the rule(35-36), now, we divide $105v$ and 103 by 3, separately and get $35v$ and 34 as quotients, and the equation to be solved is:

$$35v + 34 = 3x + 1 \text{ or } \frac{35v+33}{3} = x \quad \ldots.(6)$$

According to the rule (stated in verse 30 of GK. IX), in equation (6) the additive 33 being an exact multiple of the divisor 3, the multiplier is zero, i.e., $v = 0 = f$ and

the additive as divided by the divisor is the quotient i.e., $x = \frac{33}{3} = 11$ is the minimum solution of (6).

The general solution is: $v = 0 + 3w$ and $x = 11 + 35w$

$$N = a_1{}^2 a_2 a_3 w + a_1 a_2 a_3 f + a_1 a_2 n + a_1 m + r_1$$

$$N = 3^2 \cdot 5 \cdot 7 \cdot w + 3 \cdot 5 \cdot 7 \cdot 0 + 3 \cdot 5 \cdot 6 + 3 \cdot 4 + 1$$
$$= 315w + 103 \quad \ldots\ldots\ldots (7),$$

where w is any integer,
will also be the general solution of (1).

Again as stated in the R.35-36, we divide $315w$ and 103 by 5, separately and get $63w$ and 20 as the quotients,
Solve : $63w + 20 = 5y + 3$ or $\frac{63w+17}{5} = y \quad \ldots\ldots (8)$

5) 63 (12
60
3) 5 (1
3
2) 3 (1
2
1

12	12	12	12× 34+17=425
1	1	1× 17+17=34	34
1	1× 17+0=17	17	-
17	17	-	-
0	-	-	-

$$425 = 63 \times 6 + 47; \quad 34 = 5 \times 6 + 4$$

The minimum solution of (8) is
$w = 5 - 4 = 1 = g$; $y = 63 - 47 = 16$
The general solution is : $w = 1 + 5k$; $y = 16 + 63k$.

Therefore,
$$N = a_1{}^2 a_2{}^2 a_3 k + a_1{}^2 a_2 a_3 g + a_1 a_2 a_3 f + a_1 a_2 n + a_1 m + r_1$$
$$N = 3^2 \cdot 5^2 \cdot 7 \cdot k + 3^2 \cdot 5 \cdot 7 \cdot 1 + 3 \cdot 5 \cdot 7 \cdot 0 + 3 \cdot 5 \cdot 6 + 3 \cdot 4 + 1$$
$$N = 1575k + 418$$

Again as stated in the R.35-36, we divide $1575k$ and 418 by 7, separately and get $225k$ and 59 as the quotients,
Solving ,

$$225k + 59 = 7z + 5 \quad \text{or} \quad \frac{225k + 54}{7} = 7z \quad \ldots . (9)$$

7) 225 (32
224
1

32	32 × 54 + 0 = 1728
54	54
0	-

1728 = 225 × 7 + 153 ; 54 = 7× 7 + 5

we get, $k = 7 - 5 = 2 = h$; $z = 225 - 153 = 72$
as its minimum solution of (9), and
the general solution is: $k = 2 + 7j$; $z = 72 + 225j$
where j is any integer.
Then,
$$N = a_1{}^2 a_2{}^2 a_3{}^3 j + a_1{}^2 a_2{}^2 a_3 h + a_1{}^2 a_2 a_3 g + a_1 a_2 a_3 f + a_1 a_2 n + a_1 m + r_1$$
$$N = 3^2 \cdot 5^2 \cdot 7^2 j + 3^2 \cdot 5^2 \cdot 7 \cdot 2 + 3^2 \cdot 5 \cdot 7 \cdot 1 + 3 \cdot 5 \cdot 7 \cdot 0 + 3 \cdot 5 \cdot 6 + 3 \cdot 4 + 1$$
$$= 111025j + 3568.$$
$$\therefore N = 111025j + 3568$$
is the general solution of (1).

(14). अपि च ।

कौ रामेषुहतौ शराद्रिविहृतावेकद्विकाग्रौ तयो–
विश्लेषश्चतुराहतो नवहृतः पञ्चाग्रको जायते ।
योगोऽपि त्रिगुणश्च सायकहृतोद्व्यग्रः फलैक्यं दशा–
ऽभ्यस्तं रुद्रहृतं नगाग्रकमभूद्राशी सखे तौ वद ॥३४॥

EX.14 : "Two quantities when multiplied by 3 and 5 , and (then) divided by 5 and 7, leave 1 and 2 as remainders, (in order). Their difference when multiplied by 4 and (then) divided by 9, leaves 5 as the remainder. Their sum when multiplied by 3 and (then) divided by 5, leaves 2 as the remainder. The sum of

(the above quotients), when multiplied by 10 and (then) divided by 11, leaves 7 as the remainder. O friend, tell them." ॥34॥

That is, to find the value of x and y such that,

$3x = 5r + 1$, ……….. …. ….. ….. ………..(1a)

$5y = 7s + 2$, ……. …. . ……… ……….. …..(1b)

$4(y - x) = 9t + 5$, … … … ….. …. …. ……(1c)

$3(x + y) = 5v + 2$, … …. … ….. ……. .. …(1d)

$10(r + s + t + v) = 11w + 7$, …. …. ……….(1e)

where, x, y, r, s, t, v and w are integers.

Solution : The solution of the problem as solved by Nārāyaṇa Paṇḍita is on the following lines.

Solving the eqution (1a),

$$5r + 1 = 3x \quad \text{or} \quad \frac{5r+1}{3} = x$$

3) 5 (1
3
2) 3 (1
2
1

1	1	1 × 1 + 1 = 2
1	1 × 1 + 0 = 1	1
1	1	-
0	-	-

2 = 0 × 5 + 2 ; 1 = 0 × 3 + 1

$$\therefore x = 2 + 5k \quad \ldots\ldots\ldots(2a)$$

and $r = 1 + 3k$ …….. $(2c)$

where k is any integer.

Similarly, solving $\frac{7s+2}{5} = y$ …(1b)

5) 7 (1
5
2) 5 (2
4
1

1	1	1× 4+2 = 6
2	2× 2+0 = 4	4
2	2	-
0	-	-

6 = 0 × 7 + 6 ; 4 = 0 × 5 + 4

$$\therefore y = 6 + 7p \quad \ldots\ldots\ldots.(2b)$$

and $s = 4 + 5p$ ………..$(2d)$

where p is any integer.

Now, if we take $k = p$, relations (2a) and (2c) transform into

$$x = 2 + 5p \quad \ldots\ldots\ldots(2g)$$

and $r = 1 + 3p$ ………..$(2h)$

respectively.

Also, then the values of x, y, r and s given by relations $(2g)$, $(2b)$, $(2h)$, and $(2d)$ respectively, will satisfy both the equations (1a), and (1b).

Again substituting the values of y and x got from(2g) and (2b), in (1c) ,we obtain

$$4\{(6 + 7p) - (2 + 5p)\} = 9t + 5$$

$$4(4 + 2p) = 9t + 5$$

$$\therefore 9t - 11 = 8p \quad \text{or} \quad \frac{9t-11}{8} = p$$

8) 9 (1
8
1

1	1× 11+0=11
11	11
0	-

11 = 1 × 9 + 2 ; 1 = 1 × 8 + 3

$$t = 3 + 8q \quad \ldots\ldots\ldots(3e)$$

$$p = 2 + 9q \quad \ldots\ldots.(3f)$$

where q is any integer.

Putting the value of p in $(2g)$, $(2h)$ $2b)$, and ($(2d)$ they transform into

$x = 2 + 5p$

$= 2 + 5(2 + 9q) = 12 + 45q \quad \ldots\ldots(3a)$

$r = 1 + 3p$

$= 1 + 3(2 + 9q) = 7 + 27q \quad \ldots\ldots(3c)$

$y = 6 + 7p$

$= 6 + 7(2 + 9q) = 20 + 63q \quad \ldots\ldots(3b)$

$s = 4 + 5p$

$= 4 + 5(2 + 9q) = 14 + 45q \quad \ldots\ldots(3d)$

Now the values of x, y, r and s given by relations $(3a), (3c)$, $(3b)$, and $(3d)$ respectively, will satisfy all the three equatiions viz, (1a), (1b), and (1c).

Again substituting the values of x and y got from (3a) and (3b) in (1d), we get

$$3(x + y) = 5v + 2$$

$$3\{(12 + 45q) + (20 + 63q)\} = 5v + 2$$

$$3\{32 + 108q\} = 5v + 2$$

$$5v - 94 = 324q \quad \text{or} \quad \frac{5v-94}{324} = q$$

324) 5 (0
0
5) 324 (64
320
4) 5 (1
4
1

0	0	0	0× 6110+94=94
64	64	64× 94 + 94 = 6110	6110
1	1× 94 + 0 = 94	94	-
94	94	-	-
0	-	-	-

$94 = 18 \times 5 + 4$; $6110 = 18 \times 324 + 278$

$\therefore \quad q = 4 + 5m$; and $= 278 + 324m \quad \ldots(4f)$

where m is any integer.

Substituting this value of q in $(3a), (3b)(3c), (3d)$ and $(3e)$ they transform into

$x = 12 + 45q$

$= 12 + 45(4 + 5m) = 192 + 225m \quad \ldots(4a)$

$y = 20 + 63q$

$= 20 + 63(4 + 5m) = 272 + 315m \;\ldots\; (4b)$

$r = 7 + 27q$

$= 7 + 27(4 + 5m) = 115 + 135m \;\ldots\; (4c)$

$s = 14 + 45q$

$= 14 + 45(4 + 5m) = 194 + 225m \;\ldots(4d)$

$t = 3 + 8q$

$= 3 + 8(4 + 5m) = 35 + 40m \quad \ldots\ldots(4e)$

$v = 278 + 324m \quad \ldots\ldots\ldots(4f)$

Also the values of x, y, r, s and t given by relations $(4a), (4b), (4c), (4d)$, and $(4e)$ respectively, will satisfy all the four equations viz., $(1a), (1b), (1c)$, and $(1d)$.

Finally, substituting the values of r, s, t and v given by $(4c), (4d)$, $(4e)$ and $(4f)$ in $(1e)$

$10\{(115 + 135m) + (194 + 225m) + (35 + 40m) + (35 + 40m)\} = 11w + 7$

$$10\{622 + 724m\} = 11w + 7$$

$$11w - 6213 = 7240m \quad \text{or} \quad \frac{11w-6213}{7240} = m$$

7240) 11 (0
0
11) 7240 (658
7238
2) 11 (5
10
1

0	0	0	0+31065=31065
658	658	658× 31065+6213=2044983	20446983
5	5× 6213+0=31065	31065	-
6213	6213	-	-
0	-	-	-

$$31065 = 2824 \times 11 + 1\,;$$

$$20446983 = 2824 \times 7240 + 1223$$

$$\therefore m = 1 + 11n\,;\; w = 1223 + 7240n$$

where, n is any integer.

Putting this value of m, in $(4a),(4b),(4c),(4d),(4e)$ and $(4f)$ they transform into

$x = 192 + 225m$
$= 192 + 225(1 + 11n) = 417 + 2475n \ldots(5a)$
$y = 272 + 315m$
$= 272 + 375(1 + 11n) = 587 + 3465n \ldots\ldots(5b)$
$r = 115 + 135m$
$= 115 + 135(1 + 11n) = 250 + 1485n \ldots(5c)$
$s = 194 + 225m$
$= 194 + 225(1 + 11n) = 419 + 2475n \ldots\ldots\ldots(5d)$
$t = 35 + 40m$
$= 35 + 40(1 + 11n) = 75 + 440n \ldots\ldots\ldots(5e)$
$v = 278 + 324m$
$= 278 + 324(1 + 11n) = 602 + 3564n \ldots\ldots (5f)$

With this values of x, y, r, s, t and v gien by $(5a),(5b),(5c),(5d),(5e)$, and $(5f)$ respectively, all the equations $(1a),(1b),(1c),(1d)$, and $(1e)$ are satisfied simultaneously.

By taking $n = 0, 1,$ and 2 in succession, we get.

n	0	1	2	....
x	417	2892	5367	....
y	587	4052	7517	....

2.14.5 : EQUAL REMAINDERS :

सूत्रम् ।

तुल्येऽग्रेऽग्रं राशिः प्रक्षेपः कृतसमानहारः स्यात् ।३६(iii)।

R. 36(iii) : " In the case of equal remainders, the remainder is the number and the least common multiple (L.C.M.) of the divisors is the additive." ॥36(iii)॥

According to the rule, if,

$$N = ax + c = by + c = dz + c = \ldots.$$

then, $N = c$ with the L.C.M. of $a, b, d \ldots.$, as additive.

i.e., $N = c + pm,$

where p is an integer, and $m =$ L.C.M. of $a, b, d \ldots$

(15). उदाहरणम् ।

राशिः सखे सागरतर्कनागरन्धैर्विभक्तोऽपि निरग्रकः स्यात् ।
रूपाग्रको वा युगलाग्रको वा राशिं समाचक्ष्व तमाशु मे त्वम् ॥३५॥

EX. 15 : "O friend, tell me quickly the numbers, (which when) divided (separately) by 4, 6, 8, or 9, either leave no remainder or leave 1 as remainder or 2 as the remainder." ॥35॥ That is to find,

$N = 4x_1 + c = 6x_2 + c = 8x_3 + c = 9x_4 + c,$

when $c = 0, c = 1, c = 2.$

Solution : Comparing the equation to be solved with its general form,

$$N = a_1x_1 + c = a_2x_2 + c = a_3x_3 + c = a_4x_4 + c\,.$$

Here $a_1 = 4, a_2 = 6, a_3 = 8, a_4 = 9$

$\therefore L.C.M. of\, a_1,\ a_2,\ a_3,\ a_4$ is 72 .

According to the rule stated in verse 36(iii),

(i). $N = 0 + 72p$ when $c = 0$

(ii). $N = 1 + 72p$ when $c = 1$

(iii). $N = 2 + 72p$ when $c = 2$

where p is an integer.

Appendix - II

(Word Numerals)

A system of expressing numbers by means of words arranged as in the place-value notation is called as Word Numeral System. In this system the numerals are expressed by names of things, beings or concepts, which, naturally or in accordance with the teaching of *Śāstras,*connote *numbers.* The following is a list of words used in this system to denote numbers in Nārāyaṇa Paṇḍita's *Bījagaṇitāvataṃsa.*

Numbers	Words
0	खः , अभ्रं .
1	इन्दु , महि ,रूपं ; शशिन् .
2	अक्षि, अश्विन्, नयनं, नेत्रं, भुजः, यमल, युग्म; लोचनं.
3	अग्निः, गुणः, दहन्, राम .
4	अब्धिः .
5	इषुः , बाणः , शरः .
6	अङ्ग ; रसः .
7	नगः
8	इभः, भुजंगः, वसु ; मङ्गल .
10	ककुभ् , दिश् ,
12	अर्कः, रविः .
13	विश्व.
14	इन्द्रः , मनुः, पुरन्दरः .
15	तिथिः .
18	स्मृतिः .
20	नखः .
24	सिद्ध .
32	दन्तः

Appendix-III

Word Numerals in alphabetical order with references.

Word	Numeral	Reference Ex. number	Page
अक्षि	2	19	
अग्निः	3	18, 29	
अब्धिः	4	18	
अभ्रं	0	29	
अर्कः	12	Part II, Ex.1.	
अश्विन्	2	29	
अंग	6	19	
इन्दुः	1	18	
इन्द्रः	14	29	
इभः	8	17, 18	
इषुः	5	18, 19	
ककुभ्	10	25	
खः	0	18, 29	
गुणः	3	21	
तिथिः	15	23, 25	
दन्तः	32	17, 19	
दहन्	3	18	
दिश्	10	29	
नखः	20	16	
नगः	7	21	
नयनं	2	24,29	
नेत्रं	2	22	

पुरंदरः	14	21	
प्रकृतिः	21	21	
बणः	5	18	
भुजः	2	21	
भुजंगः	8	17	
मनुः	14	24,25	
यमल	2	17, 18	
युग्मः	2	9	
रविः	12	25	
रसः	6	21, 22	
राम	3	19	
रूपं	1	19	
लोचनं	2	18	
वसु	8	18, 22	
विश्व	13	21, 25	
शरः	5	16, 18	
शशिन्	1	18	
सिद्ध	24	15	
स्मृतिः	18	17,18	

APPENDIX - IV

Concordance of rules parallel to those of Nārāyaṇa Paṇḍita's *Bījagaṇitāvataṃsa* found in other Hindu works

Ch.	Verse number in NBi.	Cross references of rules in other Hindu works	Present Book p.no.	f.n.
		Parta I		
1.	**षट्त्रिंशत् परिकर्माणि**			
	8	(i) BrSpSi.xviii-30; (ii) GSS, i.50-51; (iii) Si.Se.xiv-3; iii.-28; (iv) BBi, R.3; Cf. HHM. II. p. 20. f.n. 4 and p.21. f.n. 1-5.,	3	2
	9(i)	(i).BrSpSi.Xviii.31-32; ii).GSS. i.-51; (iii).Si.Se.xiv-3;(iv).BBi.R.॥७॥5॥. Cf. HHM. II. p. 22.	4	3
	9(ii)	(i). BrSpSi. Xviii.-33; (ii).GSS. i.-50; (iii).Si.Se.xiv-4; (iv).BBi. R.१/7; Cf. HHM. II. pp. 22-23.	5	4
	9(iii)	(i). BrSpSi. Xviii.34; (ii).GSS.i.-50; (iii).Si.Se. xiv-4; (iv).BBi.R. ११/7. Cf. HHM. II. p. 23.	6	5
	10	(i). BrSpSi. Xviii.35; (ii).GSS.i.-52; (iii).Si.Se. xiv-5; (iv). BBi. R. ॥१३॥10॥. ; Cf. HHM. II. p. 24.	7	6
	11	. BBi. १६/12. Cf. HHM, I p.242.	8	1
	12-13	L(ASS)- 45-46 ; Cf. HHM. I. p. 242	10	2-3
	15	BBi. २०/16.	12	5
	17-18	BBi. R. ११/17; Cf. HHM, II, pp.18-19.	20	1
	19	(i). BrSpSi.xviii-41; (ii). BBi. ॥२२॥18 ॥; Cf. HHM. II. pp. 25-26.	21	2
	21-22	Cf. BrSpSi.xviii.-42. BBi. R. २६/21. HHM II. p.26, f.n.4 and 5	23	3-4
	23	BBi. vs. ॥२१॥24 ॥, HHM II. p.27, f.n.2 and 3	24	5
	24	BBi. ॥३१॥26 ॥ ; Cf. HHM II. p.28, f.n.1	25	6
	28	Cf. BBi. vs.॥३४(ii) ॥30 ॥; *(IJHS)* **28** (3), 1993. pp. 254-255.	35	2
	29	Cf. BBi. vs. ॥ ३४(i) ॥ 29 ॥	36	3
	30	Cf. Br. Sp. Si. xviii. ; GSS. vii. $88\frac{1}{2}$.	36	4
	32	Cf. BBi. ॥३७॥ 33 ॥	40	5
	36	Cf. BBi. R. ॥३९/36 ॥; IJHS, **28**(3), 1993, p. 258, f.n.22-23.	48	7
	37-38	. Cf. Br,Sp.Si. XViii, 39. IJHS, **28**(3), 1993, p. 258, f.n.16-18.	50	8
	41-43(i)	Cf. BBi. vs.44-45(i). IJHS, **28**(3), p. 260.	57	10
	43(ii)-44	Cf. (i).BrSpSi. xviii-40; (ii).SiSe. xiv-12. (iii). BBi.॥४१॥39–40॥; IJHS, **28**(3), 1993 pp. 258-259,f.n.26.	58	11
	45	Cf. BBi. R. ॥४४/44-45॥. IJHS, **28**(3), 1993, pp. 262-263,f.n.30.	58	12
	46-49	IJHS, **28**(3), 1993, p. 263, f.n.38.	60	14
	50	Cf. BBi. R. ॥४४/44-45॥. IJHS, **28**(3), 1993, p. -263, f.n.37	61	15
	51	Cf. BBi. ॥४४॥ 46-47॥; *IJHS.* **28** (3), 1993. p. 260-261.	79	16
	52	Cf. BBi. ॥४२॥ 41॥ *IJHS.* **28** (3), 1993. p. 259. (f.n. 27.)	86	20
2	**कुट्टकः**			
	53	(i). SiSe. xiv-26.; (ii).L(ASS).Vs.242(ii). (iii). BBi..R.53(i). (iv).G.K.ix.19(ii).; (v).SiTVi.xiii.p.179f; Cf. HHM. II. pp. 92- 93.	94	1
	54	Cf. (i). MSi. Xviii-1(i); (ii).G.K. ix-20(ii).; (iii). BBi.I . R.53(ii).	94	2
	55-56	Cf. (i).BBi.R. 55-56(i). (ii).G.K.ix.21-22.	95	3

	57-58(i)	Cf. (i). BBi. R.56(ii). (ii).G.K. ix.2324(i).	95	4
	58(ii)	Cf. (i). L(ASS).Vs.250.; (ii).G.K. ix-24(ii); (iii). BBi.R.59(i).	96	5
	59(i)	Cf. (i). L(ASS).Vs.-256. (ii).G.K. ix-25(i). (iii).BBi.R.64.	96	6
	59(iii)	Cf. (i). L(ASS).Vs.-256. (ii).G.K. ix-25(i). (iii).BBi.R.64.Cf.	96	7
	60	Cf. (i). L(ASS). Vs.248. ; (ii).G.K. ix-26. (iii). BBi. R. ॥५३॥ 58 ॥	106	18
	61	Cf. (i). L(ASS).Vs.252 (ii-iii).(ii). G.K. IX.-27. (iii). BBi.R.॥५६॥ 61॥ ; HHM. II. p. 112.	115	21
	$61\frac{1}{2}$	Cf. G.K. IX.-28(i).	115	22
	62	(i). BBi. R. 67(ii). (ii). G.K. ix.R. 28-29. HHM. II. pp. 121- 122.	121	25
	63	Cf. (i). L(ASS).Vs.-254. (ii).G.K. ix-30. (iii)BBi. R ॥५८॥63॥.	124	29
	64	(i). MBh. i.45.(ii). BrSpSi. Xviii.9-11. (iii). (ASS).Vs.257. (iv).BBi..R.॥६६॥71॥ (v). G.K. ix-31.	128	33
	66-67	(i).L(ASS).Vs.258.; (ii).BBi.R.72. (iii). G.K. ix. 37(ii)-39.	129	34
3. वर्गप्रकृतिः				
	70	(i). BBi. R.75 ; (ii). SiTvi. xiii. vs.209. Cf.HHM. II. P. 144.	139	1
	72-73(i)	Cf. (i) .Br.Sp.Si. xviii.64-65. (ii).BBi. R.॥७१॥77-78॥ (iii). SiTVi. Xiii.210-214. HHM. II. pp. 147- 148.	143	5
	73(i)-74(i)	Cf. (i) .Br.Sp.Si. xviii.64-65. ; (ii).BBi. R.॥७१॥77-78॥; (iii). SiTVi. Xiii.210-214. HHM. II. pp. 147- 148.	144	6
	74(ii)-75(i)	BBi. R.॥७२॥ 79 ॥ ; GK, X, R. 5(ii)-6(i); HHM. II. p. 151.	147	10

	75(ii)-76(i)	Cf. (i).BBi. R.॥७३॥ 80-81॥ ; (ii).GK.. X. 6(ii)-7(i); (iii).SiTVi.,Xiii.216. HHM. II. p.154.	151	13
	76(ii)	Cf. BBi. R. ॥७३॥81॥ ; Cf. GK. x. 6b -7a. HHM. II. p.150. (Also see f.n. 2).	152	14
	77- 80	Cf. (i).BBi..R.॥७५॥83-86॥ (ii) G.K. Ch. X. R.8-11. ; HHM. II. p. 162-166.	160	3
	81	Cf. (i). BBi, R. ॥७७॥ 88 ॥; (ii). GK. X. 12. HHM. II. pp. 178.	174	13
	82	Cf. (i). BBi. R. ॥८२॥ 93॥. ; (ii) GK.Ch.x.R.13.; HHM, II, p. 176.	180	1
	83	Cf. (i). BBi. R.॥८४॥95॥; (ii). GK. x. R. 14. ; HHM. II. p.177.	182	2
	84	Cf. (i) BBi. R. ॥८१॥ 92॥ ; (ii)GK.Ch.x. R. 16. ;(iii). Also see, Br.Sp.Si. xviii-66. HHM, II, p. 174.	187	5
	86	Cf. GK. Ch.X. R.17. [GK, Part II, p. 244]	192	12
Parta II				
	3-4	(i). Br.Sp.Si.xviii. 43. Pr. Comm.. ; (ii). NBi. II. R. 3-4.(Gloss) Cf. HHM. II. p.29.	198	1
	5-6(i)	(i). Ā. ii. 30. ; (ii). Br.Sp.Si.xviii. 43. ; (iii). BBi. R. 101. Cf. HHM. II. p. 41.	199	2
	6(ii)-7	BBi. ॥८१॥102 ॥ ; HHM. II. pp. 126 - 127.	200	3

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ABOUT THE BOOK

The present work contains the Sanskrit text of *Bījagaṇitāvataṃsa of* Nārāyaṇa Paṇḍita with Introduction, English translation, exposition, notes, rationales of the rules, and complete solution of illustrative examples according to the methods of Nārāyaṇa using modern symbols.

The use of symbols - letters of the alphabet to denote unknowns - and equations are the foundations of the science of algebra. The Hindus were the first to make systematic use of the letters of the alphabet to denote unknowns. They were also the first to classify and make a detailed study of equations, Thus they may be said to have given birth to the modern science of algebra.

We are indebted to the Hindus for the technique and fundamental results of algebra just as we owe to them the place-value system in arithmetic, and invention of the basic function 'sine' in Trigonometry.

A study of this book, *Bījagaṇitāvataṃsa,* will reveal to the reader the remarkable progress in algebra made by the Hindus at an early date.